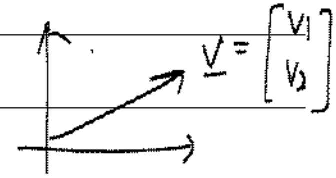
Ch 6 Orthogonality.

Sec. 6.1. Geometry of vectors.

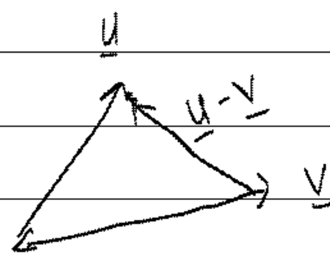
<u>Defn</u> Let $\underline{v} \in \mathbb{R}^n$. The norm (length) of $\underline{v}$ is defined as

$$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

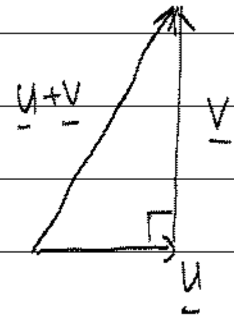


$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2}$

- $\|\underline{v}\| = 0$ iff $\underline{v} = \underline{0}$
- If $\|\underline{w}\| = 1$, then $\underline{w}$ is called a unit vector.
- For any nonzero $\underline{u}$, $\dfrac{\underline{u}}{\|\underline{u}\|}$ is a unit vector
- Distance between two vectors $\underline{u}$ and $\underline{v}$ is defined as $\|\underline{u} - \underline{v}\|$



Pythagorean Thm in $\mathbb{R}^2$.



$\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ iff $\underline{u}$ and $\underline{v}$ are perpendicular.

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\|\underline{u}+\underline{v}\|^2 = (u_1+v_1)^2 + (u_2+v_2)^2 = \overbrace{u_1^2+u_2^2}^{\|\underline{u}\|^2} + \overbrace{v_1^2+v_2^2}^{\|\underline{v}\|^2} + 2(u_1v_1 + u_2v_2)$$

$\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ iff $u_1v_1 + u_2v_2 = 0$.

- We can conclude from Pythagorean thm that $\underline{u}$ and $\underline{v}$ are perpendicular iff $u_1v_1+u_2v_2 = 0$.

<u>Defn</u> Dot Product

Let $\underline{u}$ and $\underline{v} \in \mathbb{R}^n$.

$$\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Defn. We say $\underline{u}$ and $\underline{v}$ are orthogonal (or perpendicular) if $\underline{u} \cdot \underline{v} = 0$.

Ex. $\underline{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ $\underline{w} = \begin{bmatrix} -8 \\ 3 \\ 2 \end{bmatrix}$

$\underline{u} \cdot \underline{v} = (2)(1) + (-1)(4) + (3)(-2) = -8$.

$\underline{v} \cdot \underline{w} = (1)(-8) + (4)(3) + (-2)(2) = 0$. $\underline{v}$ and $\underline{w}$ are orthogonal

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Claim 1 $\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = \underline{v}^T \underline{u}$.

$$\underline{u}^T \underline{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

#.

Claim 2 Let A be $m \times n$, B be $n \times m$, $\underline{u}$ be $n \times 1$, $\underline{v}$ be $m \times 1$

(a) $(A\underline{u}) \cdot \underline{v} = \underline{u} \cdot (A^T \underline{v})$ (b) $\underline{u} \cdot (B\underline{v}) = (B^T \underline{u}) \cdot \underline{v}$

Pf. $(A\underline{u}) \cdot \underline{v} \underset{\text{Claim 1}}{=} (A\underline{u})^T \underline{v} = \underline{u}^T (A^T \underline{v}) \underset{\text{Claim 1}}{=} \underline{u} \cdot (A^T \underline{v})$ #.

Thm 6.1. Let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$, c is a scalar. Then the following are true.

(a) $\underline{u} \cdot \underline{u} = \|\underline{u}\|^2$ (b) $\underline{u} \cdot \underline{u} = 0$ iff $\underline{u} = \underline{0}$.

(c) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ (d) $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$.

(e) $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$

(f) $(c\underline{u}) \cdot \underline{v} = c(\underline{u} \cdot \underline{v}) = \underline{u} \cdot (c\underline{v})$

(g) $\|c\underline{u}\| = |c| \, \|\underline{u}\|$.

$$\|2\underline{u} + 3\underline{v}\|^2 = (2\underline{u} + 3\underline{v}) \cdot (2\underline{u} + 3\underline{v}) = 2\underline{u} \cdot 2\underline{u} + 2\underline{u} \cdot 3\underline{v} + 3\underline{v} \cdot 2\underline{u} + 3\underline{v} \cdot 3\underline{v}$$
$$= 4\|\underline{u}\|^2 + 9\|\underline{v}\|^2 + 12\, \underline{u} \cdot \underline{v}.$$

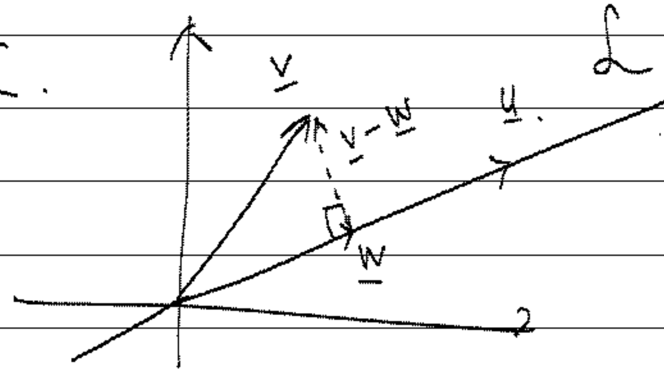
$$(2x + 3y)^2 = 4x^2 + 9y^2 + 12xy.$$

Thm 6.2. Pythagorean Thm in $\mathbb{R}^n$.

Let $\underline{u}$ and $\underline{v} \in \mathbb{R}^n$. Then $\underline{u}$ and $\underline{v}$ are orthogonal iff $\|\underline{u} + \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$.

Pf $\|\underline{u} + \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\underline{u} \cdot \underline{v} = \|\underline{u}\|^2 + \|\underline{v}\|^2$ iff $\underline{u} \cdot \underline{v} = 0$.

Orthogonal Projection of $\underline{v}$ onto a line $\mathcal{L}$.
(OP)



Let $\underline{w}$ be the OP of $\underline{v}$ onto $\mathcal{L}$.

Let $\underline{u}$ be a nonzero vector on $\mathcal{L}$

Then $\underline{w} = c\underline{u}$ for some scalar c.

$(\underline{v} - \underline{w})$ and $\underline{u}$ are orthogonal.

$(\underline{v} - \underline{w}) \cdot \underline{u} = 0$.

$(\underline{v} - c\underline{u}) \cdot \underline{u} = 0 \qquad \underline{v} \cdot \underline{u} = c\, \underline{u} \cdot \underline{u}$.

$$\Rightarrow \quad c = \frac{\underline{v} \cdot \underline{u}}{\|\underline{u}\|^2}$$

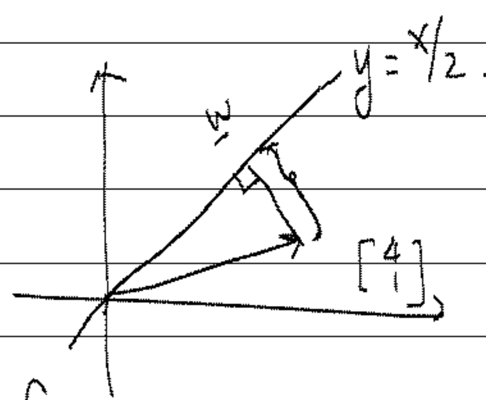
The OP of $\underline{v}$ onto $\mathcal{L}$ $(\underline{u})$ is

$$\boxed{\underline{w} = \frac{\underline{v} \cdot \underline{u}}{\|\underline{u}\|^2}\, \underline{u}}$$

Let $\underline{u}' = \dfrac{\underline{u}}{\|\underline{u}\|}$ (unit vector), then $\underline{w} = (\underline{v} \cdot \underline{u}')\, \underline{u}'$

Ex. Find the distance from $\begin{bmatrix}4\\1\end{bmatrix}$ to $y = x/2$.

Let $\underline{w}$ be the OP of $\begin{bmatrix}4\\1\end{bmatrix}$ onto $\mathcal{L}$



distance $\overset{\Delta}{=} \left\| \begin{bmatrix}4\\1\end{bmatrix} - \underline{w} \right\|$

choose $\underline{u} = \begin{bmatrix}2\\1\end{bmatrix}$ be a vector on $\mathcal{L}$.

The OP of $\begin{bmatrix}4\\1\end{bmatrix}$ onto $\begin{bmatrix}2\\1\end{bmatrix}$ is $\underline{w} = \dfrac{\begin{bmatrix}4\\1\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix}}{\left\| \begin{bmatrix}2\\1\end{bmatrix} \right\|^2} \begin{bmatrix}2\\1\end{bmatrix}$

$$\underline{w} = \frac{9}{5} \begin{bmatrix}2\\1\end{bmatrix}.$$

distance $= \left\| \begin{bmatrix}4\\1\end{bmatrix} - \frac{9}{5}\begin{bmatrix}2\\1\end{bmatrix} \right\| = \frac{2}{5}\sqrt{5}$

Cauchy - Schwarz Inequality.

Let $\underline{u}, \underline{v} \in \mathbb{R}^n$. Then $|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \, \|\underline{v}\|$

Pf. If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$, then it is trivially true.

Assume $\underline{u} \neq \underline{0}$ and $\underline{v} \neq \underline{0}$.

define $\underline{u}' = \dfrac{\underline{u}}{\|\underline{u}\|}$ and $\underline{v}' = \dfrac{\underline{v}}{\|\underline{v}\|}$. $\|\underline{u}'\| = \|\underline{v}'\| = 1$.

$$0 \le \|\underline{u}' \pm \underline{v}'\|^2 = \|\underline{u}'\|^2 + \|\underline{v}'\|^2 \pm 2\,\underline{u}' \cdot \underline{v}'$$

$$= 2 \pm 2\,\underline{u}' \cdot \underline{v}'$$

$$\mp 2\,\underline{u}' \cdot \underline{v}' \le 2.$$

$$|\underline{u}' \cdot \underline{v}'| \le 1.$$

$$\left| \frac{\underline{u}}{\|\underline{u}\|} \cdot \frac{\underline{v}}{\|\underline{v}\|} \right| \le 1 \Rightarrow |\underline{u} \cdot \underline{v}| \le \|\underline{u}\|\,\|\underline{v}\| \quad \#$$

- In P88, $|\underline{u} \cdot \underline{v}| = \|\underline{u}\|\,\|\underline{v}\|$ iff $\underline{u} = c\underline{v}$ or $\underline{v} = c\underline{u}$ for some scalar c.

- Ex. $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

$$|u_1 v_1 + u_2 v_2 + u_3 v_3| \le \sqrt{u_1^2 + u_2^2 + u_3^2}\,\sqrt{v_1^2 + v_2^2 + v_3^2}$$

Triangle Inequality.

$$\|\underline{u} \pm \underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|.$$

pf. $\|\underline{u} \pm \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 \pm 2\underline{u} \cdot \underline{v}$

$$\le \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2|\underline{u} \cdot \underline{v}|$$

$$\le \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2\|\underline{u}\|\,\|\underline{v}\| = (\|\underline{u}\| + \|\underline{v}\|)^2$$

$$\|\underline{u} \pm \underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$$

#.

Sec. 6.1. T&F, 47, 48, 88, 89, 93, 97, 98.

Quiz 2 5/23 9:20 ~ 10:10 am

Ch5, Sec. 6.1 and Sec. 6.2.

Sec. 6.2. Orthogonal Vectors.

<u>Defn</u>. A subset of $\mathbb{R}^n$ is called an orthogonal set if any pair of distinct vectors are orthogonal.

i.e. $S = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ orthogonal if $\underline{v}_i \cdot \underline{v}_j = 0$ for any $i \neq j$.

<u>Defn</u>. Orthonormal set = orthogonal set + unit vectors.

<u>Defn</u>. Let V be a subspace of $\mathbb{R}^n$.

S is called an orthogonal (orthonormal) basis for V

if (i) S is a basis for V

(ii) S is orthogonal (orthonormal).

Ex. $S = \{\underline{v}\}$ orthogonal

Ex. $S = \{\underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

Ex. $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$ orthogonal $\underline{v}_i \cdot \underline{v}_j = 0$.

$S' = \left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \right\}$ orthonormal.

$S\ (S')$ is an orthogonal (orthonormal) basis for $\mathbb{R}^3$.

Thm 6.5. Any orthogonal set of nonzero vectors is l. indep

Pf. Let $S = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ be orthogonal and $\underline{v}_i \neq \underline{0}$.

Suppose there exist $c_1, c_2, \cdots, c_k$ such that

$$c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_k \underline{v}_k = \underline{0}. \qquad \text{———} (1)$$

$(1) \cdot \underline{v}_1$, we get

$$c_1 (\underline{v}_1 \cdot \underline{v}_1) + c_2 \cancelto{0}{(\underline{v}_2 \cdot \underline{v}_1)} + \cdots + c_k \cancelto{0}{(\underline{v}_k \cdot \underline{v}_1)} = \underline{0} \cdot \underline{v}_1 = 0$$

$$c_1 \|\underline{v}_1\|^2 = 0 \Rightarrow c_1 = 0.$$

$(1) \cdot \underline{v}_2$, we get

$$c_1 (\underline{v}_1 \cdot \underline{v}_2) + c_2 (\underline{v}_2 \cdot \underline{v}_2) + \cdots + c_k (\underline{v}_k \cdot \underline{v}_2) = \underline{0} \cdot \underline{v}_2 = 0 \Rightarrow c_2 = 0.$$

$\vdots$

$c_1 = 0,\ c_2 = 0,\ \cdots,\ c_k = 0 \Rightarrow S$ is l. indep #

Representation of a vector in terms of orthogonal/orthonormal basis.

Let $S = \{\underline{v_1}, \underline{v_2}, \cdots, \underline{v_k}\}$ be a basis for V.

For any $\underline{v} \in V$, there exist $c_1, c_2, \cdots, c_k$ s.t.

$$\underline{v} = c_1 \underline{v_1} + c_2 \underline{v_2} + \cdots + c_k \underline{v_k} \quad \text{——— (2)}$$

$$[\underline{v_1}\ \underline{v_2}\ \cdots\ \underline{v_k}] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \underline{v}.$$

- We need to solve a system of l. eqns to get $c_1, \cdots, c_k$.

- Now suppose that S is an orthogonal basis for V.

$(2) \cdot \underline{v_1}$, we get

$$\underline{v} \cdot \underline{v_1} = c_1 \underline{v_1} \cdot \underline{v_1} + c_2 \cancelto{0}{\underline{v_2} \cdot \underline{v_1}} + \cdots + c_k \cancelto{0}{\underline{v_k} \cdot \underline{v_1}}$$

$$c_1 = \frac{\underline{v} \cdot \underline{v_1}}{\|\underline{v_1}\|^2}.$$

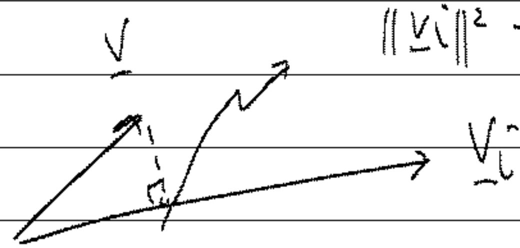
(2) $\cdot \underline{v}_2$, we get

$$c_2 = \frac{\underline{v} \cdot \underline{v}_2}{\|\underline{v}_2\|^2}.$$

$$\boxed{\underline{v} = \frac{\underline{v} \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 + \frac{\underline{v} \cdot \underline{v}_2}{\|\underline{v}_2\|^2}\underline{v}_2 + \cdots + \frac{\underline{v} \cdot \underline{v}_k}{\|\underline{v}_k\|^2}\underline{v}_k}$$

OP of $\underline{v}$ onto $\underline{v}_1$ OP of $\underline{v}$ onto $\underline{v}_2$

$$\frac{\underline{v} \cdot \underline{v}_i}{\|\underline{v}_i\|^2}\underline{v}_i$$



Let $\underline{w}_i = \frac{\underline{v}_i}{\|\underline{v}_i\|}$, then

$$\underline{v} = (\underline{v} \cdot \underline{w}_1)\,\underline{w}_1 + (\underline{v} \cdot \underline{w}_2)\,\underline{w}_2 + \cdots + (\underline{v} \cdot \underline{w}_k)\,\underline{w}_k$$

Thm 6.6. Gram-Schmidt Process.

Let $S = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$ be a l. indep subset of $\mathbb{R}^n$.

Let $S' = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k\}$, where $\underline{v}_1 = \underline{u}_1$, and

for $i \geq 2$,

$$\underline{v}_i = \underline{u}_i - \frac{\underline{u}_i \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 - \frac{\underline{u}_i \cdot \underline{v}_2}{\|\underline{v}_2\|^2} - \cdots - \frac{\underline{u}_i \cdot \underline{v}_{i-1}}{\|\underline{v}_{i-1}\|^2}\underline{v}_{i-1}$$

Then S' is an orthogonal basis for Span S.

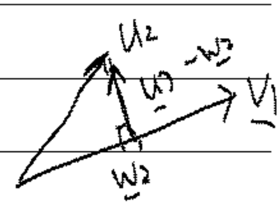
If V = Span S, S is a basis for V.

S' is an orthogonal basis for V.

$$\underline{v}_1 = \underline{u}_1$$

OP of $\underline{u}_2$ onto $\underline{v}_1$.

$$\underline{v}_2 = \underline{u}_2 - \left(\frac{\underline{u}_2 \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1\right)$$



$$\underline{v}_3 = \underline{u}_3 - \frac{\underline{u}_3 \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 - \frac{\underline{u}_3 \cdot \underline{v}_2}{\|\underline{v}_2\|^2}\underline{v}_2.$$

OP of $\underline{u}_3$ onto $\underline{v}_1$ OP of $\underline{u}_3$ onto $\underline{v}_2$

$\{\underline{v_1}\}$ is an orthogonal basis for $\mathrm{Span}\{\underline{u_1}\}$

$\{\underline{v_1}, \underline{v_2}\}$ - - - - - - - - - $\mathrm{Span}\{\underline{u_1}, \underline{u_2}\}$.

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ - - - - - - - $\mathrm{Span}\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$

$\underline{v_1}$ is a l.c. of $\underline{u_1}$

$\underline{v_2}$ - - - - - of $\underline{u_1}, u_2$

$\underline{v_3}$ - - - - - - - of $\underline{u_1}, \underline{u_2}, \underline{u_3}$

<u>Ex</u>. Let $V = \mathrm{Span}\, S$ and $S = \{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$ where

$\underline{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\underline{u_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\underline{u_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$. Find an orthogonal basis and an orthonormal basis for V.

$\underline{v_1} = \underline{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\|\underline{v_1}\|^2 = 4$.

$$\underline{v_2} = \underline{u_2} - \frac{\underline{u_2} \cdot \underline{v_1}}{\|\underline{v_1}\|^2} \underline{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}. \quad \|\underline{v_2}\|^2 = 2.$$

$$\underline{v}_3 = \underline{u}_3 - \frac{\underline{u}_3 \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 - \frac{\underline{u}_3 \cdot \underline{v}_2}{\|\underline{v}_2\|^2}\underline{v}_2 .$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} .$$

$$\underline{v}_3 = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \|\underline{v}_3\|^2 = \frac{1}{4} .$$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for V.

$\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is an orthonormal basis for V.

Proof of Gram-Schmidt Process. Thm 6.6.

- Proof by induction on the number of vectors k.

$\underline{k=1}$. $S_1' = \{\underline{v}_1\}$ is an orthogonal basis for $\mathrm{Span}\{\underline{u}_1\}$. So the thm is true for $k = 1$.

Assume it is true for $k = i-1$. We need to prove that it is true for $k = i$.

Assume $S'_{i-1} = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{i-1}\}$ is an orthogonal basis for $\underbrace{\text{Span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_{i-1}\}}_{S_{i-1}}$.

Look at $k = i$.

$$S'_i = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{i-1}, \underline{v}_i\}, \text{ where}$$

$$\underline{v}_i = \underline{u}_i - \frac{\underline{u}_i \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 - \dots - \frac{\underline{u}_i \cdot \underline{v}_{i-1}}{\|\underline{v}_{i-1}\|^2}\underline{v}_{i-1} \quad \text{--- (3)}$$

<u>Claim 1</u> $\underline{v}_i$ is orthogonal to $\underline{v}_1, \dots, \underline{v}_{i-1}$ and $\underline{v}_i \neq \underline{0}$.

(3) $\cdot \underline{v}_1$, we get

$$\underline{v}_i \cdot \underline{v}_1 = \underline{u}_i \cdot \underline{v}_1 - \frac{\underline{u}_i \cdot \underline{v}_1}{\cancel{\|\underline{v}_1\|^2}}\cancel{\underline{v}_1 \cdot \underline{v}_1} - \frac{\underline{u}_i \cdot \underline{v}_2}{\|\underline{v}_2\|^2}\underbrace{\cancel{\underline{v}_2 \cdot \underline{v}_1}}_{0} - \dots$$

$$= 0 \qquad - \frac{\underline{u}_i \cdot \underline{v}_{i-1}}{\|\underline{v}_{i-1}\|^2}\underbrace{\cancel{\underline{v}_{i-1} \cdot \underline{v}_1}}_{0}.$$

(3) $\cdot \underline{v}_2$, we get

$$\underline{v}_i \cdot \underline{v}_2 = \underline{u}_i \cdot \underline{v}_2 - \frac{\underline{u}_i \cdot \underline{v}_1}{\|\underline{v}_1\|^2}\underline{v}_1 \cdot \underline{v}_2 - \frac{\underline{u}_i \cdot \underline{v}_2}{\|\underline{v}_2\|^2}\underline{v}_2 \cdot \underline{v}_2 - \dots - \frac{\underline{u}_i \cdot \underline{v}_{i-1}}{\|\underline{v}_{i-1}\|^2}\underline{v}_{i-1} \cdot \underline{v}_2 = 0$$

$\therefore$ $\underline{v_i}$ is orthogonal to $\underline{v_1}, \underline{v_2}, \dots, \underline{v_{i-1}}$.

$\underline{v_i} \neq \underline{0}$? If $\underline{v_i} = \underline{0}$, then (3) implies $\underline{u_i}$ is a l.c. of $\underline{v_1}, \dots, \underline{v_{i-1}}$.

i.e. $\underline{u_i}$ is a l.c. of $\underline{u_1}, \dots, \underline{u_{i-1}}$!

a contradiction ($\because$ $\underline{u_i}$ is l. indep).

So $\underline{v_i} \neq \underline{0}$.

$S_i' = \{\underline{v_1}, \dots, \underline{v_i}\}$ is an orthogonal set of nonzero vectors.

$\because$ $\underline{v_1}, \dots, \underline{v_i}$ are l.c. of $\underline{u_1}, \underline{u_2}, \dots, \underline{u_i}$,

Span S_i' is a subspace of Span S_i

$S_i = \{\underline{u_1}, \dots, \underline{u_i}\}$.

$\dim(\text{Span } S_i') = i = \text{Span } S_i$.

From Thm 4.9, we conclude that Span S_i' = Span S_i.

S_i' is an orthogonal basis for Span S_i.

The thm is true for $k = i$. #.

Sec. 6.2. T&F, 14, 15, 16, 55, 56, 57.

Quiz 2 5/23, 9:20 ~ 10:10 am Ch5, 6.1, 6.2.

QR Factorization* (NOT in any exam).

Let $m\times 3$ $A = \begin{bmatrix} \underline{a_1} & \underline{a_2} & \underline{a_3} \end{bmatrix}$ and rank $A = 3$.

Then $A = QR$ where $m\times 3$ Q: has orthonormal col. vectors,
3×3 R: is an upper triangular matrix.

Pf: $\underbrace{\{\underline{a_1}, \underline{a_2}, \underline{a_3}\}}_{\text{l. indep.}} \xrightarrow{\text{G. S.}} \underbrace{\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}}_{\text{orthogonal.}} \xrightarrow[\text{norm}]{\text{divide}} \underbrace{\{\underline{w_1}, \underline{w_2}, \underline{w_3}\}}_{\text{orthonormal.}}$

$\underline{a_1} \in \text{Span}\{\underline{a_1}\} = \text{Span}\{\underline{w_1}\}$. ① — $\underline{a_1} = r_{11}\underline{w_1}$

$\underline{a_2} \in \text{Span}\{\underline{a_1}, \underline{a_2}\} = \text{Span}\{\underline{w_1}, \underline{w_2}\}$ ② — $\underline{a_2} = r_{12}\underline{w_1} + r_{22}\underline{w_2}$

$\underline{a_3} \in \text{Span}\{\underline{a_1}, \underline{a_2}, \underline{a_3}\} = \text{Span}\{\underline{w_1}, \underline{w_2}, \underline{w_3}\}$ ③ — $\underline{a_3} = r_{13}\underline{w_1} + r_{23}\underline{w_2} + r_{33}\underline{w_3}$

$$A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} = \begin{bmatrix} \underline{w}_1 & \underline{w}_2 & \underline{w}_3 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = QR.$$

r_{23} (3) $\cdot \underline{w}_2$ $\quad \underline{w}_2 \cdot \underline{a}_3 = r_{23}$,

$$r_{ij} = \underline{w}_i \cdot \underline{a}_j$$