Ch 3 Determinants (Square Matrices).

Sec. 3.1 Co-factor Expansion

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $BA=(ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

if $(ad-bc)\neq 0$, then $\frac{1}{ad-bc} BA = I_2$.

- If $ad-bc\neq 0$, then A is invertible.
- If $ad-bc=0$, then A is not invertible.

Defn $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$.

Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $n\times n$

Defn A_{ij} is the $(n-1)\times(n-1)$ submatrix of A obtained by removing the i^{th} row and the j^{th} col. of A.

$A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix}$ $A_{23}=\begin{bmatrix} 1 & 2 \\ 7 & 9 \end{bmatrix}$, $A_{31}=\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

<u>Defn</u> Determinant of A $n \times n$.

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

Co-factor Expansion.

Ex.

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix} = 1 \cdot \det \underbrace{\begin{bmatrix} 5 & 6 \\ 9 & 8 \end{bmatrix}}_{A_{11}} - 2 \cdot \det \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$$

$$= 1 \cdot (5 \cdot 8 - 6 \cdot 9) - 2 \cdot (4 \cdot 8 - 6 \cdot 7) + 3 \cdot (4 \cdot 9 - 5 \cdot 7)$$

$$= 9$$

Defn The (i,j)th co-factor of $A = \underbrace{(-1)^{i+j} \det A_{ij}}_{C_{ij}}$.

- Using co-factor, $\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$

Thm 3.1. Co-factor expansion along the i^{th} row.

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}.$$

(Proof in reference [4]).

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix}$.

along 3rd row

$\det A = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$.

$= 7 (-1)^{3+1} \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} + 9 \cdot (-1)^{3+2} \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} + 8 \cdot (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$

$= 9$.

- From Thm 3.1, if A has a zero row, then $\det A = 0$.

Ex. $\det \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ \vdots & & & \vdots \\ & & & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} (-1)^{1+1} \det \begin{bmatrix} a_{22} & 0 & \cdots & 0 \\ a_{32} & a_{33} & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ a_{n2} & \cdots & & a_{nn} \end{bmatrix} + 0 + \cdots + 0$.

lower triangular matrix

$= a_{11} \cdot a_{22} (-1)^{1+1} \det \begin{bmatrix} a_{33} & 0 & \cdots 0 \\ & \ddots & 0 \\ a_{n3} & & a_{nn} \end{bmatrix} + 0 \cdots + 0$.

$\cdots = a_{11} a_{22} \cdots a_{nn}$.

Similarly, if A is an upper triangular matrix,

then $\det A = a_{11} a_{22} \cdots a_{nn}$.

Sec. 3.2. Properties of Determinants.

Thm 3.3.

Type 1 (a) If $A \xrightarrow{r_i \leftrightarrow r_j} B$, then $\det B = -1 \cdot \det A$.

Type 2 (b) If $A \xrightarrow[k \neq 0]{k r_i} B$, then $\det B = k \cdot \det A$.

Type 3. (c) If $A \xrightarrow{c r_i + r_j \to r_j} B$, then $\det B = \det A$.

(d) $\det(EA) = \det E \cdot \det A$.

- see text ,ok for a proof.

Some observations.

(1) Extend pa. (d) to $\det(E_k E_{k-1} \cdots E_1 A)$

$= \det E_k \det E_{k-1} \cdots \det E_1 \det A$.

(2) If A has two identical rows $(r_i \,\&\, r_j)$, then $\det A = 0$

$A \xrightarrow{r_i \leftrightarrow r_j} B = A$ from Thm 3.3(a) $\det B = (-1)\det A$.

$\det B = \det A$.

(3). $\det(kA) = k^n \det A$.

$A \xrightarrow{kr_1} \xrightarrow{kr_2} \cdots \xrightarrow{kr_n} kA$.

(4) If $r_j = c r_i$, then $\det A = 0$.

$A \xrightarrow{-cr_i + r_j \to r_j} B$ the j^{th} row of B is a zero row.

$0 = \det B = \det A$

(5). determinant of el matrices $I_n \xrightarrow{e.r.o} E$

Type 1. $I_n \xrightarrow{r_i \leftrightarrow r_j} E_1$, $\det E_1 = (-1) \cdot \det I_n = -1$.

Type 2 $I_n \xrightarrow[k \neq 0]{kr_i} E_2$, $\det E_2 = k \cdot \det I_n = k \neq 0$.

Type 3 $I_n \xrightarrow{cr_i + r_j \to r_j} E_3$, $\det E_3 = \det I_n = 1$.

$\det E \neq 0$ for all el. matrices.

Forward pass of G.E.

$$A \xrightarrow[\text{G.E.}]{\text{FP of}} U \quad \text{r.e.f is an upper triangular matrix.}$$

- In FP, we use only Type 1 and Type 3 e.r.o's.

$$A \sim \sim \cdots \sim U \text{ r.e.f.}$$

use Type 1 e.r.o's "ℓ times"

$$\det A = (-1)^{\ell} \det U$$

Ex.
$$A = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{bmatrix} \xrightarrow[2r_1+r_4\to r_4]{r_1 \leftrightarrow r_3} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 0 & -4 & 12 & 1 \end{bmatrix}$$

$$\xrightarrow[4r_2+r_4\to r_4]{r_2 \leftrightarrow r_3} \begin{bmatrix} \boxed{-2} & 0 & 4 & -7 \\ 0 & \boxed{1} & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 24 & -11 \end{bmatrix} \xrightarrow{-6\cdot r_3+r_4\to r_4} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$\det A = (-1)^2 \cdot \det U = -8.$$

Sec. 3.1 20, 27, 65, 77, 80.

Sec. 3.2. 21, 24, 33, 34, 72~75, 78.

Review 27, 28, 33~40.

Thm 3.4. A, B $n\times n$ The following are true.

(a) A is invertible if and only if $\det A \neq 0$.

(b) $\det(AB) = \det A \cdot \det B$.

(c) $\det A^T = \det A$

(d) If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

Proof: Suppose A is invertible

From Thm 2.6, $A = E_k E_{k-1} \cdot E_2 E_1$

(a) $\det A = \det(E_k \cdots E_2 E_1) = \det E_k \cdots \det E_2 \det E_1 \neq 0$.

(b) $\det(AB) = \det(E_k \cdots E_2 E_1 B) = \det E_k \cdots \det E_2 \cdot \det E_1 \det B$

$= \det(E_k \cdot E_{k-1} \cdots E_2 E_1) \det B = \det A \det B$.

(c) $A^T = E_1^T E_2^T \cdots E_k^T$

- Notice that E_i^T are also el. matrices of the same type.

ith $\rightarrow$ $\begin{pmatrix} 1 & & \\ c & \ddots & \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & c & \\ & \ddots & \\ & & 1 \end{pmatrix}$ $\leftarrow$ jth row, ith col.

($\uparrow$ jth)

- In other word, $\det E_i^T = \det E_i$.

$$\det A^T = \det (E_1^T E_2^T \cdots E_k^T)$$
$$= \det E_1^T \det E_2^T \cdots \det E_k^T$$
$$= \det E_1 \det E_2 \cdots \det E_k.$$
$$= \det E_k \cdots \det E_2 \cdot \det E_1$$
$$= \det (E_k E_{k-1} \cdots E_2 E_1) = \det A.$$

(d). $A^{-1} A = I_n$. $1 = \det I_n = \det(A^{-1}A) = \det A^{-1} \det A$ ($\because A^{-1}$ is invertible.)

$$\Rightarrow \det A^{-1} = \frac{1}{\det A}.$$

Suppose A is not invertible.

- There exists invertible P s.t. $PA = R$ rref.
- The last row of R is a zero row $\therefore$ $\operatorname{rank} A < n$.

(a) $\det A = \det(P^{-1}R) = \det P^{-1} \det \underset{0}{\underset{\shortparallel}{R}} = 0$

(b) $\det(AB) = \det(P^{-1}RB) = \det P^{-1} \det \underset{0}{\underset{\shortparallel}{(RB)}} = 0$

The last row of RB is a zero row.

$\det A \cdot \det B = 0 \cdot \det B = 0$.

$\Rightarrow \det(AB) = \det A \cdot \det B$.

(c). A^T is also not invertible.

$\det A^T = 0 = \det A$. #

- Using $\det A^T = \det A$, we can replace "row" in the observations by "column".

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq \det A \det D - \det B \det C.$$

(A: $n\times n$, B: $n\times m$, C: $m\times n$, D: $m\times m$)

Ex. $$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \cdot \det D.$$

(A: $n\times n$, B: $n\times m$, 0: $m\times n$, D: $m\times m$)

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix}.$$

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det \begin{bmatrix} I_n & 0 \\ 0 & D \end{bmatrix} \det \begin{bmatrix} A & B \\ 0 & I_m \end{bmatrix} = \det D \cdot \det A.$$

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END of Ch 3.

Cramer's rule. $A\underline{x} = \underline{b}$ (A: $n\times n$ invertible)

$x_i = \cdots$

Ch 4. Subspaces and Their Properties.

Defn. a subset W of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if W satisfies

(i) $\underline{0} \in W$

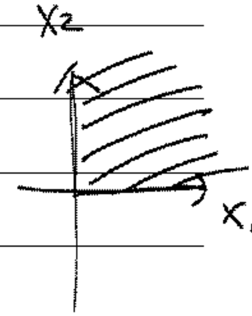
(ii) for any $\underline{u}, \underline{v} \in W$, $\underline{u}+\underline{v} \in W$.

(iii) for any $\underline{u} \in W$, $c\underline{u} \in W$ for any scalar c.

Ex 1. $\mathbb{R}^n$ is a subspace.

Ex 2 $W = \{\underline{0}\}$. zero subspace.

Ex. 3 empty set is not a subspace.



Ex 4 $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \geq 0,\ x_2 \geq 0 \right\}$

(i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$. ✓

(ii) for any $\underline{u}, \underline{v} \in W$, $\underline{u}+\underline{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$.

$u_1+v_1 \geq 0$, $u_2+v_2 \geq 0$ $\Rightarrow$ $\underline{u}+\underline{v} \in W$.

(iii) Take $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = -1$.

$c\underline{u} \notin W$.

W is not a subspace.

Ex 5 $W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; 5w_1 - 4w_2 + 2w_3 = 0 \right\}$.

(i) $\underline{0} \in W$.

(ii) if $\underline{u}$ and $\underline{v} \in W$, i.e. $5u_1 - 4u_2 + 2u_3 = 0$, $5v_1 - 4v_2 + 2v_3 = 0$.

then $\underline{u}+\underline{v} \in W$? $5(u_1+v_1) - 4(u_2+v_2) + 2(u_3+v_3) = 0$.

$\underline{u}+\underline{v} \in W$.

(iii) If $\underline{u} \in W$, then $c\underline{u} \in W$ for any scalar c.

W is a subspace.

Thm 4.1. The span of any nonempty subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Pf $S = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$ is a subset of $\mathbb{R}^n$.

(i) $\underline{0} \in \text{Span } S$ $\because$ $\underline{0} = 0 \cdot \underline{u}_1 + 0 \cdot \underline{u}_2 + \cdots + 0 \cdot \underline{u}_k$.

(ii) If $\underline{u}$ and $\underline{v} \in$ Span ?, i.e. $\underline{u} = c_1 \underline{u}_1 + \cdots + c_k \underline{u}_k$

$\underline{v} = d_1 \underline{u}_1 + \cdots + d_k \underline{u}_k$.

$$\underline{u} + \underline{v} = (c_1 + d_1)\underline{u}_1 + (c_2 \; d_2)\underline{u}_2 + \cdots + (c_k + d_k)\underline{u}_k$$

$\in$ Span S.

(iii) If $\underline{u} \in$ Span S, then $c\underline{u} \in$ Span S for any scalar c.

Span S is a subspace of $\mathbb{R}^n$.

Ex. $W = \left\{ \begin{bmatrix} 2a-3b \\ b \\ -a+4b \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; a, b \in \mathbb{R} \right\}$

$$\begin{bmatrix} 2a-3b \\ b \\ -a+4b \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$$

$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$.

Subspaces Associated with a matrix.

$$m\times n \quad A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix} = \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \vdots \\ \underline{r}_m \end{bmatrix}$$

($\underline{a}_1$: $m\times 1$; $\underline{r}_i$: $1\times n$)

$\underline{r}_i$ row vectors

$\underline{a}_i$ col. vectors.

Defn Column space of A, $\text{Col } A = \text{Span}\{\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n\}$.

Row space of A; $\text{Row } A = \text{Span}\{\underline{r}_1^T, \underline{r}_2^T, \cdots, \underline{r}_m^T\}$.

Null space of A, $\text{Null } A$ = soln set to $A\underline{x} = \underline{0}$.

- $\text{Col } A$ is a subspace of $\mathbb{R}^m$
- $\text{Row } A$ - - - - - of $\mathbb{R}^n$
- $\text{Row } A = \text{Col } A^T$.

Thm 4.2. $\text{Null } A$ is a subspace of $\mathbb{R}^n$.

Pf. (i) $\underline{0} \in \text{Null } A$ $\because A\cdot\underline{0} = \underline{0}$.

(ii) If $\underline{u}, \underline{v} \in \text{Null } A$, i.e. $A\underline{u} = 0$ & $A\underline{v} = \underline{0}$

then $\underline{u} + \underline{v} \in \text{Null } A$, $\because A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v} = \underline{0}$

(iii) $c\underline{y} \in \text{Null } A \quad \because \quad A(c\underline{y}) = cA\underline{y} = \underline{0}$

for any scalar c. //

Ex 1 $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$. Find a generating set for each of Col A, Row A, Null A respectively.

- Recall if $V = \text{Span } S$, then S is called a generating set for V.

a generating set for Col A $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix} \right\}$.

— — — — — for Row A $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix} \right\}$.

Null A $\quad A\underline{x} = \underline{0} \quad A \xrightarrow{G.E} \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rref

$\quad x_1 \quad x_2 \quad x_3 \quad x_4$.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ a generating set for Null A $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Subspaces Associated with linear transformation L.T.

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear with standard matrix A $m \times n$

$$T(\underline{x}) = A\underline{x} \text{ for all } \underline{x} \in \mathbb{R}^n.$$

$$A = \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & \cdots & T(\underline{e}_n) \end{bmatrix}.$$

- range of $T = \text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} = \text{Col } A$.
- null space of T = soln set to $A\underline{x} = \underline{0}$ = Null A.

Ex. 2. $T: \mathbb{R}^4 \to \mathbb{R}^3$ $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$.

Find a generating set for range of T and null space of T, respectively.

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$ same as ex. 1.

Sec. 4.1. T&F, 24, 32, 40, 42, 72~80, 97~100.

Sec. 4.2. Basis and Dimension.

<u>Defn</u> Let V be a nonzero subspace of $\mathbb{R}^n$.

S is called a basis for V if

(i) $\text{Span } S = V$ and (ii) S is l. indep.

- a basis for V is a l. indep generating set for V.
- basis for zero subspace is undefined in our text.
- In other texts, the basis for a zero subspace is defined as the empty set ϕ.

Ex. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

Ex. $\{ \underline{e}_1, \underline{e}_2, \cdots, \underline{e}_n \}$ - - . . . . . $\mathbb{R}^n$.

standard basis

Ex. $\left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ c_2 \end{bmatrix} \right\}$, $c_1, c_2 \neq 0$ is also a basis for $\mathbb{R}^2$.

Basis for Col A

$A = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n]$ $m \times n$.

$$\text{Col } A = \text{Span}\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}.$$

- Non pivot cols are l. c. of its preceding pivot cols. So they can be removed without the span.
- Pivot cols are l. indep.

- The pivot cols of A form a basis for Col A.

Ex. $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$.

from previous example, we know that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ is a basis for Col A.

Thm 4.3. (Reduction Thm). Let S be a generating set for a nonzero subspace V of $\mathbb{R}^n$. Then S can be reduced to a basis for V by removing some vectors from S.

Pf. Let $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$.

$V = \text{Span}\, S = \text{Col}\, A$, where $A = [\underline{u}_1 \ \underline{u}_2 \ \cdots \ \underline{u}_k]$.

All the pivot cols of A form a basis for V.

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