Review

- A is an orthogonal matrix if its cols are orthonormal.
- $\|A\underline{v}\| = \|\underline{v}\|$, $A^{-1} = A^T$

Sec. 6.6. Symmetric matrices.

$$A^T = A. \quad n \times n$$

Thm 6.14. If $\underline{u}$ and $\underline{v}$ are eigenvectors of a symmetric matrix corresponding to distinct eigenvalues, then $\underline{u}$ and $\underline{v}$ are orthogonal.

Pf: Suppose $A^T = A$! $A\underline{u} = \lambda_1 \underline{u}$, $A\underline{v} = \lambda_2 \underline{v}$, $\lambda_1 \neq \lambda_2$.

$$\underline{u}^T A \underline{v} = \underline{u}^T \lambda_2 \underline{v} = \lambda_2 \underline{u}^T \underline{v} = \lambda_2 \underline{u} \cdot \underline{v}.$$

scalar

$$(\underline{u}^T A \underline{v})^T = \underline{v}^T A^T \underline{u} = \underline{v}^T A \underline{u} = \lambda_1 \underline{v}^T \underline{u} = \lambda_1 \underline{u} \cdot \underline{v}$$

$$\lambda_1 \underline{u} \cdot \underline{v} = \lambda_2 \underline{u} \cdot \underline{v} \Rightarrow \underline{u} \cdot \underline{v} = 0 \quad (\because \lambda_1 \neq \lambda_2) \quad \#$$

Thm 6.15. A is symmetric if and only if there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

That is, A is symmetric if and only if there exist an orthogonal matrix P and a diagonal matrix D such that

$$A = PDP^T$$

Pf ($\Leftarrow$) $A^T = (PDP^T)^T = PD^TP^T = PDP^T = A$.

($\Rightarrow$) see Appendix C.

#.

Ex. $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ Find orthogonal P & diagonal D s.t. $A = PDP^T$.

① CP of $A = \det(A - tI) = -(t-2)^2(t-8)$

$\lambda_1 = 2 \quad m_1 = 2$

$\lambda_2 = 8 \quad m_2 = 1$

② Find an orthonormal basis for each eigenspace.

$\underline{\underline{E_{\lambda_1}}}$ $(A - \lambda_1 I)\underline{X} = \underline{0}$ .... $\underline{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for E_{λ_1}

· use Gram-Schmidt process and divide norm

$\underline{V}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{V}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. form an orthonormal basis for E_{λ_1}.

$\underline{\underline{E_{\lambda_2}}}$ $(A - \lambda_2 I)\underline{X} = \underline{0}$ ... $\underline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\underline{V}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms an orthonormal basis for E_{λ_2}.

$$P = \begin{bmatrix} \underline{V}_1 & \underline{V}_2 & \underline{V}_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix}}_{2\times 4} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix}}_{1\times 4} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\begin{bmatrix} e & f & g & h \end{bmatrix}$$

3×4

In general.

$$\underbrace{\begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \vdots \\ \underline{b}_n^T \end{bmatrix}}_{n\times p} = \underbrace{\underline{a}_1}_{m\times 1} \underbrace{\underline{b}_1^T}_{1\times p} + \underline{a}_2 \underline{b}_2^T + \cdots + \underline{a}_n \underline{b}_n^T \quad \text{—— (1)}$$

Spectral Decomposition of a symmetric matrix

$$A = PDP^T = \begin{bmatrix} \underline{v_1} & \underline{v_2} & \cdots & \underline{v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 \underline{e_1} & \lambda_2 \underline{e_2} & \cdots & \lambda_n \underline{e_n} \end{bmatrix} \begin{bmatrix} \underline{v_1}^T \\ \underline{v_2}^T \\ \vdots \\ \underline{v_n}^T \end{bmatrix}$$

$\underline{v_i}$: orthonormal vectors

$$A = \begin{bmatrix} \lambda_1 \underline{v_1} & \lambda_2 \underline{v_2} & \cdots & \lambda_n \underline{v_n} \end{bmatrix} \begin{bmatrix} \underline{v_1}^T \\ \underline{v_2}^T \\ \vdots \\ \underline{v_n}^T \end{bmatrix}$$

Using (1), we get

$$A = \lambda_1 \underline{v_1}\,\underline{v_1}^T + \lambda_2 \underline{v_2}\,\underline{v_2}^T + \cdots + \lambda_n \underline{v_n}\,\underline{v_n}^T$$

spectral decomposition of A

define $\underline{v_i}\,\underline{v_i}^T = P_i$.

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$$

Thm 6.16. A: $n \times n$ symmetric. Let $\{\underline{v_1}, \underline{v_2}, \cdots, \underline{v_n}\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting eigenvectors of A and the corr. eigenvalues be $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Then there exist symmetric matrices $P_1, P_2, \cdots, P_n$ such that the following are true.

(a) $A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$.

(b) rank $P_i = 1$.

(c) $P_i P_i = P_i$ (d) $P_i P_j = \Theta$ for $i \neq j$.

(e). $P_i \underline{v_i} = \underline{v_i}$ (f) $P_i \underline{v_j} = \underline{0}$ for $i \neq j$.

Pf $P_i = \underline{v_i}\,\underline{v_i}^T$ $(P_i)^T = P_i$.

$$P_i P_i = (\underline{v_i}\,\underbrace{\underline{v_i}^T)(\underline{v_i}}_{=1}\,\underline{v_i}^T) = \underline{v_i}\,\underline{v_i}^T = P_i$$

$$P_i P_j = (\underline{v_i}\,\underbrace{\underline{v_i}^T)(\underline{v_j}}_{=0}\,\underline{v_j}^T) = \Theta$$

$$P_i \underline{v_i} = (\underline{v_i}\,\underbrace{\underline{v_i}^T)\,\underline{v_i}}_{=1} = \underline{v_i}.$$

#.

Define $W_i = \mathrm{Span}\{\underline{v_i}\}$. What is ortho. proj (OP) matrix for W_i?

$C = \underline{v_i}$

$$P_{W_i} = C(C^TC)^{-1}C^T = \underline{v_i}\,\underline{v_i}^T = P_i$$

- P_i are OP matrix for $\mathrm{Span}\{\underline{v_i}\}$.

Ex 1 $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. find spectral decomposition of A.

from previous ex $\underline{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\underline{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$P_1 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_2 = \frac{1}{6} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}, \quad P_3 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A = 2P_1 + 2P_2 + 8P_3.$$

Spectral Approximation of symmetric A.

arrange eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n, \quad P_i = \underline{v}_i \underline{v}_i^T.$$

If we want approximate A by a rank-1 matrix, then.

$$A_1 = \lambda_1 P_1 \approx A.$$

If - - - - - - - - rank 2 matrix, then

$$A_2 = \lambda_1 P_1 + \lambda_2 P_2 \approx A.$$

n^2, $n+1$, $\underline{v}_1, \lambda_1$

Sec. 6.6. Exercise 10, 12, 17, 18, T&F, 47~54
59, 66, 68, 72.

Positive definite / semidefinite matrices. 59 ~ 73.

<u>Defn.</u> a symmetrix matrix A is said to be positive definite (semi).
if $\underline{v}^T A \underline{v} > 0$ for any $\underline{v} \neq \underline{0}$.
(≥ 0)

- A is positive definite (semi) iff all its eigenvalues > 0 (≥ 0).

- Let A be positive definite

$$\lambda_{max} \geq \cdots \geq \lambda_{min} > 0.$$

$$\max_{\underline{v} \neq \underline{0}} \frac{\underline{v}^T A \underline{v}}{\underline{v}^T \underline{v}} = \lambda_{max}$$

$$\min_{\underline{v} \neq \underline{0}} \frac{\underline{v}^T A \underline{v}}{\underline{v}^T \underline{v}} = \lambda_{min}.$$

END of Ch 6

Ch 7. Vector Spaces.

Defn a vector space (VS) is a set V for which two operations, called vector addition $+$ and scalar multiplication, are defined so that for any $u, v, w \in V$ and scalars a, b, $u+v$ and av are unique elements in V and such that the following axioms hold.

(1) $u+v = v+u$

(2) $(u+v)+w = u+(v+w)$

(3) There is an element 0 in V such that $u+0=u$ for each $u \in V$. (0 is called the zero vector in V)

(4) For each $u \in V$, there is an element $(-u)$ in V such that $u+(-u)=0$. ($-u$ is called the additive inverse of u)

(5) $1u = u$ scalar

(6) $(ab)u = a(bu)$

(7) $a(u+v) = au+av$.

- The elements in V are called vectors.

<u>Ex 1</u> $\mathbb{R}^n$ is a VS.

<u>Ex 2</u> $M_{m\times n}$: set of all $m\times n$ matrices.

vector addition : $A+B \in M_{m\times n}$.

Scalar mul. : $cA \in M_{m\times n}$.

$M_{m\times n}$ is a VS.

<u>Ex. 3</u> Functional Space.

Let S be a nonempty set.

$f: S \to \mathbb{R}$ is a function from S to $\mathbb{R}$.

<u>Defn</u> Function Space

$\mathcal{F}(S)$ = set of all ftns from S to $\mathbb{R}$.

- If $f \in \mathcal{F}(S)$, then f is a ftn from S to $\mathbb{R}$.

Vector addition: $(f_1 + f_2)(t) = f_1(t) + f_2(t) \in \mathcal{F}(S)$.

Scalar mul: $(cf_1)(t) = c \cdot f_1(t) \in \mathcal{F}(S)$.

zero funtion $f_0 : S \to \mathbb{R}$ $f_0(s) = 0$ for each $s \in S$.

$$(f + f_0)(t) = f(t) + f_0(t) = f(t) \text{ for each } f \in \mathcal{F}(S)$$

$\mathcal{F}(S)$ is a VS.

Ex. 4. $\mathcal{P}$: set of all polynomials.

vector addition: $p(x) + q(x) \in \mathcal{P}$.

scalar mul: $cp(x) \in \mathcal{P}$.

Zero polynomial: $P_0(x) = 0$ for any x.

$\mathcal{P}$ is a VS.

Ex 5. $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$: set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

If $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

vector addition: $(T_1 + T_2)(\underline{v}) = T_1(\underline{v}) + T_2(\underline{v}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

scalar mul: $(cT_1)(\underline{v}) = cT_1(\underline{v}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

zero transformation $T_0: \mathbb{R}^n \to \mathbb{R}^m$ $T_0(\underline{v}) = \underline{0}$ for each $\underline{v} \in \mathbb{R}^n$.

$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is a VS.

In Review Exercise P11.

V: set of all positive real numbers.

if $x \in V$, then $x > 0$.

scalar $c \in \mathbb{R}$. $u, v \in V$.

Vector addition: $u \oplus v = uv \in V$.

scalar mul : $c \odot u = u^c \in V$ for any $u \in V$ and any $c \in \mathbb{R}$.

③ $u=1$ is the zero vector in V.

$$1 \oplus v = 1 \cdot v = v \quad \text{for each } v \in V.$$

④ additive inverse of u is $1/u$

$$u \oplus 1/u = 1 \quad \text{zero vector.}$$

⑤ $1 \odot u = u^1 = u.$

⑥ $(ab) \odot u = u^{ab} = (u^b)^a = a \odot (b \odot u)$

⑦ $a \odot (u \oplus v) = (u \oplus v)^a = (uv)^a = u^a v^a = u^a \oplus v^a$

$$= a \odot u \oplus a \odot v$$

V is a VS.

Quiz 3. June 13, 9:20 am. Sec. 6.3, 6.4, 6.5, 6.6. Sec. 7.1.

Subspaces.

Let V be VS. A subset W is called a subspace of V if

(i) zero vector is in W.

(ii) For any $u, v \in W$, $u+v \in W$

(iii) For any $u \in W$, and any Scalar c, $cu \in W$.

Ex1 $M_{n\times n}$: set of all $n\times n$ matrices.

W = set of all $n\times n$ matrices with trace equal to zero.

trace $A = a_{11} + a_{22} + \cdots + a_{nn}$.

(i) $0 \in W$

(ii) If $A, B \in W$, then $\operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B$

$= 0 + 0 = 0$

$A+B \in W$.

(iii) $cA \in W$. W is a subspace

Ex-2. $V = \mathcal{F}(S)$, let $s_0 \in S$.

W = set of all functions f with $f(s_0) = 0$.

(i) $f_0(t) \in W$, $\because$ $f_0(s_0) = 0$.

(ii) If $f_1, f_2 \in W$, then $(f_1+f_2)(s_0) = f_1(s_0) + f_2(s_0) = 0$.

$$f_1 + f_2 \in W.$$

(iii) $cf_1 \in W$ $\because$ $(cf_1)(s_0) = c \cdot f_1(s_0) = 0$.

W is a subspace of $\mathcal{F}(S)$.

Ex. 3. $C(\mathbb{R})$: set of all continuous ftns from $\mathbb{R}$ to $\mathbb{R}$.

$C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$.

(i) $f_0 \in C(\mathbb{R})$ $\because$ zero ftn is continuous.

(ii) If $f_1, f_2 \in C(\mathbb{R})$, $f_1 + f_2 \in C(\mathbb{R})$

(iii) $cf_1 \in C(\mathbb{R})$

Ex 4 P_n: set of all polynomials with degree $\le n$, $n \ge 1$.

P_n is a subspace of P.

(i) $P_0(x) \in P_n$. (ii)(iii) $p(x)+q(x)$, $cp(x) \in P_n$.

Linear Combinations (l.c).

<u>Defn</u>. Let S be a nonempty subset of a VS V.

v is called a l.c. of vectors in S if there exist $v_1, v_2, \cdots, v_m \in S$ and scalars $c_1, c_2, \cdots, c_m$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

Ex. $S = \{1, x, x^2, x^3\}$. subset of P_3.

$p(x) = 2 - 3x + 5x^3$ is a l.c. of vectors in S.

Ex. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ subset of $M_{2\times 2}$.

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad a, b, c \in \mathbb{R}.$$

$= \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is a l.c. of vectors in S.

Ex. $S = \{1, x, x^2, \ldots, x^n, \ldots\}$.

all polynomials are l.c. of vectors in S.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

e^x is NOT a l.c. of vectors in S.

<u>Defn</u> The Span of S, Span S, is the set of all l.c's of vectors in S.

Ex. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Span S = set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

$a, b, c \in \mathbb{R}$.

= set of all 2×2 matrices with trace equal to 0.

Ex. $S = \{1, x, x^2, x^3\}$.

$\operatorname{Span} S = \mathbb{P}_3$

Ex. $S = \{1, x, x^2, \ldots, x^n, \ldots\}$

$\operatorname{Span} S = \mathbb{P}$.

Thm 7.3. The span of a nonempty subset S of a VS V is a subspace of V.

Sec. 7.1. T&F, 8, 13, $\underbrace{60 \sim 73,\ 88 \sim 94}_{\text{Subspaces.}}$

Quiz 3 June 13, 9:20 am.