Sec. 7.4. Matrix Representation of Linear Operators.

<u>Defn</u> $T: V \to V$ linear is called l. operator on V.

- Let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be a basis for V. Then for each $v \in V$, there are unique scalars $c_1, \cdots, c_n$ such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

- Isomorphism $\phi_{\mathcal{B}}: V \to \mathbb{R}^n$ $\phi_{\mathcal{B}}(v) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

- Defn Coordinate vector of v relative to $\mathcal{B}$.

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \left(\mathcal{B}\text{- coordinate of } v \right)$$

$\mathbb{R}^2$ $\mathcal{B} = \{e_1, e_2\}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Inverse $\phi_{\mathcal{B}}^{-1}: \mathbb{R}^n \to V$ $\phi_{\mathcal{B}}^{-1}\left([v]_{\mathcal{B}}\right) = v$.

$$T: V \to V.$$

$$\phi_{\mathcal{B}}: V \to \mathbb{R}^n$$

- Composite $\phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a l. operator on $\mathbb{R}^n$.

$$\mathbb{R}^n \xrightarrow{\phi_{\mathcal{B}}^{-1}} V \xrightarrow{T} V \xrightarrow{\phi_{\mathcal{B}}} \mathbb{R}^n.$$

$$\phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1} (\quad)$$

- Let A $(n \times n)$ be the standard matrix of $\phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1}$.

- <u>Defn</u>. A is called the matrix repr. of T with respect to $\mathcal{B}$, denoted by $[T]_{\mathcal{B}}$.

(B - matrix). $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$.

- Claim $[T]_{\mathcal{B}} = \left[[T(v_1)]_{\mathcal{B}} \; [T(v_2)]_{\mathcal{B}} \; \cdots \; [T(v_n)]_{\mathcal{B}} \right]$.

<u>Pf</u>: Note $v_j = 0 v_1 + 0 v_2 + \cdots 1 v_j + 0 \cdots + 0 v_n$.

$$\phi_{\mathcal{B}}(v_j) = [v_j]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow j^{th} \text{ entry} = \underline{e}_j.$$

$\phi_{\mathcal{B}}^{-1}(\underline{e}_j) = v_j$

Let A be the standard matrix of $\phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1}$.

Look at the jth col of A

$$A\,\underline{e}_j = \phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1}(\underline{e}_j) = \phi_{\mathcal{B}} T(v_j) = [T(v_j)]_{\mathcal{B}}.$$

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<u>Ex 1</u> $T: P_2 \to P_2$ $T(p(x)) = p(0) + 3p'(1)x + p(2)x^2$.

$\mathcal{B} = \{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}\}$ is a basis for P_2. Find $[T]_{\mathcal{B}}$.

$$T(v_1) = 1 + 3\cdot \underbrace{0}_{p'(1)}\, x + 1\, x^2 \qquad [T(v_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(v_2) = 0 + 3\ \underbrace{1}_{(x)'|_{x=1}}\, x + 2\, x^2 \qquad [T(v_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}.$$

$$T(v_3) = 0 + 3\cdot 2\, x + 4x^2 \qquad [T(v_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 1 & 2 & 4 \end{bmatrix}.$$

Ex 2. $U: M_{2\times 2} \to M_{2\times 2}$ $U(A) = A^T$. Find $[U]_\beta$.

$$\beta = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4.}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}.$$

$U(v_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = v_1 + 0\cdot v_2 + 0\cdot v_3 + 0 v_4$, $[U(v_1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$U(v_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = v_3$ $\qquad [U(v_2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\qquad [U]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$U(v_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = v_2$ $\qquad [U(v_3)]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

$U(v_4) = v_4$ $\qquad [U(v_4)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Defn $T: V \to V$ linear.

a nonzero vector $v \in V$ is said to be an eigenvector of T if $T(v) = \lambda v$ for some scalar λ. eigenvalue λ.

Ex 3 : $\mathcal{D} : C^\infty \to C^\infty$. $\mathcal{D}(f) = f'$

$\mathcal{D}(e^{at}) = a e^{at}$ for any real number a.

- any real number is an eigenvalue of $\mathcal{D}$ and any exponential ftn is an eigenvector of $\mathcal{D}$.

$\mathcal{D}$ has infinitely many eigenvalues and infinitely many l. indep eigenvectors.

Ex. 4 $U : M_{2\times2} \to M_{2\times2}$ $U(A) = A^T$.

For Any nonzero symmetric matrix A, $U(A) = A^T = A$

A is an eigenvector of U with $\lambda = 1$.

For any nonzero skew-symm matrix A, $U(A) = A^T = -A$

$(A^T = -A)$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. A is an eigenvector of U with $\lambda = -1$.

(P42, U has only two eigenvalues $\lambda = +1$, and $\lambda = -1$.)

Claim Let $T: V \to V$ linear. and $\mathcal{B}$ be a finite basis for V.

Let A be the standard matrix of $\phi_{\mathcal{B}} T \phi_{\mathcal{B}}^{-1}$, i.e. $A = [T]_{\mathcal{B}}$.

Then the following are true.

(a) v is an eigenvector of T iff $[v]_{\mathcal{B}}$ is an eigenvector of A.

(b) v is in the null space (or range of T) iff

$[v]_{\mathcal{B}}$ is in the null space (or col space) of A.

(c) T is invertible iff A is invertible. Moreover

$$[T^{-1}]_{\mathcal{B}} = A^{-1}.$$

Ex 1 <u>conti</u> find a basis for null space of T and range of T.

$\mathcal{B} = \{1, x, x^2\}$.

$$A = [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 6 \\ 1 & 2 & 4 \end{bmatrix},$$

$\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ form a basis for Null A.

$$\phi_B^{-1}\left(\begin{bmatrix}0\\-1\\2\end{bmatrix}\right) = 0\cdot 1 + (-1)X + 2X^2 = -X + 2X^2$$

$\{-X+2X^2\}$ form a basis for the null space of T.

$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\3\\2\end{bmatrix}\right\}$ form a basis for Col A.

$\{1+X^2,\ 3X+2X^2\}$ for a basis for the range of T.

Ex. 2 conti Find a basis for each eigenspace of U.

$$\mathcal{B} = \left\{ \underset{V_1}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}, \underset{V_2}{\begin{bmatrix}0&1\\0&0\end{bmatrix}}, \underset{V_3}{\begin{bmatrix}0&0\\1&0\end{bmatrix}}, \underset{V_4}{\begin{bmatrix}0&0\\0&1\end{bmatrix}} \right\}. \quad [U]_{\mathcal{B}} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} = A.$$

CP of $A = (t+1)(t-1)^3$. $\lambda_1 = -1$, $m_1 = 1$

$\lambda_2 = 1$, $m_2 = 3$.

<u>E_{λ_1}</u> ...... $\left\{\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right\}$ form a basis for E_{λ_1}.

$$\phi_B^{-1}\left(\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right) = 0V_1 + 1\cdot V_2 + (-1)V_3 + 0\cdot V_4 = \begin{bmatrix}0&1\\-1&0\end{bmatrix}.$$

U has 2 eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$.

$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ form a basis for E_{λ_1} of U.

<u>E_{λ_2} of A</u> ..... $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ form a basis for E_{λ_2} of A.

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ form a basis for E_{λ_2} of U.

Exercise Sec. 7.4. T&F, 11, 12, 16, 26, 45.

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51 ~ 53. $T: \underset{\mathcal{B}}{V} \to \underset{\mathcal{C}}{W}$ linear.

$[T]_{\mathcal{B}}^{\mathcal{C}}$

期末考 6/27 (三) 9:20 ~ 11:00am

Ch 5, Ch 6, Sec. 7.1, 7.2, 7.3, 7.4.

July 2 (一)

- 12:20 ~ 13:00 Check exam paper at BL 113.
- 11 am anounce final scores on web.
- 2 pm announce final grade.
- No change after 5 pm.

Sec. 7.5 Inner Product Space.

Defn. An inner product on a VS V is a real-valued function, that assigns any pair of vectors u and v a scalar, denoted by $\langle u, v \rangle$, such that for any $u, v, w \in V$ and scalar c, the following are satisfied.

(1) $\langle u, u \rangle > 0$ for any $u \neq \theta$

(2) $\langle u, v \rangle = \langle v, u \rangle$.

(3) $\langle u+w, v \rangle = \langle u, v \rangle + \langle w, v \rangle$.

(4) $\langle cu, v \rangle = c \langle u, v \rangle$.

- a vector space with an inner product is called an inner product space.

Defn Norm $\|v\| = \sqrt{\langle v, v \rangle}$.

unit vector $\|v\| = 1$.

distance btw u and v = $\|u - v\|$.

Ex. 1 $V = \mathbb{R}^n$. $\underline{u} \cdot \underline{v}$ is also an inner product.

Ex. 2. $V = C([a,b])$. $\underline{u} \cdot \underline{v} = u_1 v_1^* + u_2 v_2^* + \cdots + u_n v_n^*$.

$$\langle f, g \rangle = \int_a^b f(t) g(t)\, dt.$$

(2) $\langle g, f \rangle = \int_a^b g(t) f(t)\, dt = \int_a^b f(t) g(t)\, dt = \langle f, g \rangle$.

(3) $\langle f_1 + f_2, g \rangle = \int_a^b (f_1 + f_2)\, g\, dt = \int_a^b f_1 g\, dt + \int_a^b f_2 g\, dt$

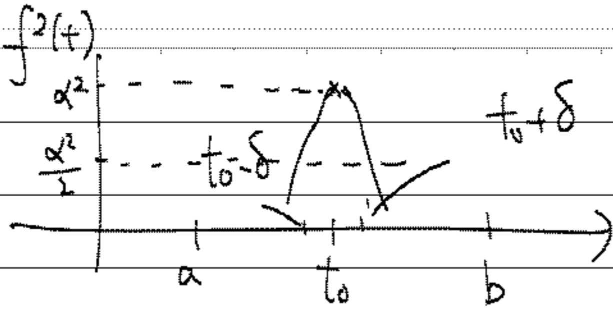
$= \langle f_1, g \rangle + \langle f_2, g \rangle$.

(4) $\langle cf, g \rangle = c \langle f, g \rangle$.

(1) If $f(t) \neq \theta(t)$ in $t \in [a,b]$, then $f(t) \neq 0$ for some $t = t_0 \in [a,b]$.

$f^2(t)$ is also continuous in $[a,b]$. $f(t_0) = \alpha \neq 0$.

$f^2(t_0) = \alpha^2 > 0$



$$f^2(t) > \frac{\alpha^2}{2} \quad \text{for} \quad t \in [t_0-\delta,\ t_0+\delta].$$

$\exists\, \delta > 0$ such that whenever $|t-t_0| < \delta$, $f^2(t) > \alpha^2/2$.

$$\langle f, f\rangle = \int_a^b f^2(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} f^2(t)\,dt.$$

$$> \int_{t_0-\delta}^{t_0+\delta} \alpha^2/2 \, dt = 2\delta \cdot \frac{\alpha^2}{2} = \delta\alpha^2 > 0.$$

$\int_a^b f(t)\,g(t)\,dt$ is an inner product.

Ex. 3. $V = M_{m\times n}$. $\langle A, B\rangle = \operatorname{trace}(AB^T)$

Frobanius inner product

In P74, $\operatorname{trace} AB^T = a_{11}b_{11} + a_{12}b_{12} + \cdots + a_{1n}b_{1n} + a_{21}b_{21} + \cdots + a_{mn}b_{mn}$

(1) $\langle A, A\rangle = a_{11}^2 + \cdots + a_{1n}^2 + \cdots + a_{mn}^2 > 0 \quad \because$ some $a_{ij} \neq 0$.

$A \neq 0$

(2). $\langle A, B\rangle = \langle B, A\rangle$.

(3) $\langle A_1 + A_2, B\rangle = \langle A_1, B\rangle + \langle A_2, B\rangle$.

(4) $\langle cA_1, B\rangle = c\langle A_1, B\rangle$.

$$\langle A, B\rangle = \operatorname{trace}(AB^T) \text{ is an inner product.}$$

In fact $\operatorname{trace}(AB^T) = \operatorname{trace}(BA^T) = \operatorname{trace}(A^T B) = \operatorname{trace}(BA^T)$

<u>Defn</u> u and v are orthogonal if $\langle u, v\rangle = 0$.

orthogonal set, orthogonal basis, orthonormal basis.

Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V.

For any $v \in V$, we have

$$v = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + \cdots + \langle v, v_n\rangle v_n$$

Gram-Schmidt Process.

$$S = \{u_1, \cdots, u_k\} \text{ basis for } V \xrightarrow{G.S.} S = \{v_1, v_2, \cdots, v_k\}$$

orthogonal basis for V.

$v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 .$$

Ex. $M_{2\times 2}$ $\quad \mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_4} \right\}$.

$\langle v_i, v_j \rangle = 0$, $\|v_i\| = 1$, $\mathcal{B}$ is an orthonormal basis for $M_{2\times2}$.

Ex. $\mathcal{P}_2$, $\mathcal{B} = \{\overset{u_1}{1}, \overset{u_2}{x}, \overset{u_3}{x^2}\}$

$$\langle p_1(x), p_2(x) \rangle = \int_{-1}^{1} p_1(x)\, p_2(x)\, dx .$$

$$\langle u_1, u_3\rangle = \int_{-1}^{1} 1 \cdot x^2\, dx = 2/3 \neq 0 .$$

Use GS process

$$V_1 = u_1 = 1, \quad \|v_1\|^2 = \langle v_1, v_1\rangle = \int_{-1}^{1} 1 \cdot 1\, dx = 2 .$$

$$V_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} V_1 = X - \frac{\langle x, 1\rangle}{2} 1 = X.$$

$$\langle x, 1\rangle = \int_{-1}^{1} 1 \cdot x\, dx = 0$$

$$\|V_2\|^2 = \int_{-1}^{1} x \cdot x\, dx = 2/3.$$

$$V_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= x^2 - \frac{\langle x^2, 1\rangle}{2} 1 - \frac{\langle x^2, x\rangle}{2/3} x. = x^2 - 1/3.$$

$$\langle x^2, 1\rangle = 2/3, \quad \langle x^2, x\rangle = \int_{-1}^{1} x^2 x\, dx = 0$$

$$\|V_3\|^2 = \int_{-1}^{1} (x^2 - 1/3)^2\, dx = 8/45.$$

$\left\{ \frac{1}{\sqrt{2}}, \frac{x}{\sqrt{\frac{2}{3}}}, \frac{x^2 - 1/3}{\sqrt{\frac{8}{45}}} \right\}$ form an orthonormal basis for P_2.

Normalized Legendre Polynomial.

Orthogonal complement $S^\perp$, $W^\perp$.

Let W be a subspace of V. Then for any $v \in V$, there is a unique $w \in W$ and $z \in W^\perp$ such that $v = w + z$.

Let $\{w_1, \dots, w_k\}$ be an orthonormal basis for W, then $w = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \cdots + \langle v, w_k \rangle w_k$.

w is the OP of v onto W.

- w is the vector in W that is closest to v.