- Reduced Row Echelon Form (rref).

- Gaussian Elimination (G.E.).

- A procedure for solving $A\underline{x} = \underline{b}$.

(1) Form $[A\ \underline{b}]$

(2) Apply G.E $[A\ b] \rightsquigarrow [R\ \underline{c}]$ rref.

(3) Solve $R\underline{x} = \underline{c}$.

Ex. 1. $A\underline{x} = \underline{b}$

$$[A\ \underline{b}] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 1 & 8 & 0 \\ 1 & -2 & -1 & 5 \\ 1 & 1 & 5 & -1 \end{bmatrix} \xrightarrow[\substack{-1r_1+r_3 \to r_3 \\ -1r_1+r_4 \to r_4}]{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 3 & 6 & -6 \\ 0 & -1 & -2 & 2 \\ 0 & 2 & 4 & -4 \end{bmatrix}$$

pivot cols.

$$\xrightarrow[-\frac{2}{3}r_2+r_4 \to r_4]{\frac{1}{3}r_2+r_3 \to r_3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 6 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[r_2+r_1 \to r_1]{\frac{1}{3}r_2 \to r_2} \begin{array}{c} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & c_i \end{array} \end{array} = \underset{\text{rref}}{[R\ \underline{c}]}$$

basic var's   free var.

The soln in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

Some Observations $[A\ \underline{b}] \rightsquigarrow [R\ \underline{c}]$ rref.

- $A\underline{x} = \underline{b}$ is consistent if and only if $[R\ \underline{c}]$ does not have a row $[0\ 0 \cdots 0\ 0\ 1]$.
- For a consistent $A\underline{x} = \underline{b}$, it has infinitely many solns iff it has a free var.

- Pivot cols of rref. are standard vectors $\underline{e}_1, \underline{e}_2, \cdots$

<u>Defn</u>. rank / nullity of $\underset{m\times n}{A}$.

m rows
n cols. $\underset{m\times n}{A} \rightsquigarrow R$ rref.

rank A = # of nonzero rows in R.

nullity $A = n - \text{rank } A$.

<u>Ex 1</u> rank $[A\ \underline{b}] = 2$, nullity $[A\ \underline{b}] = 4 - 2 = 2$.

rank $A = 2$, nullity $A = 3 - 2 = 1$.

- rank A = # of basic var's = # of pivot cols.
- nullity A = # of free var's.

- $\operatorname{rank} A \le m$, $\operatorname{rank} A \le n$.
- $0 \le \operatorname{nullity} A \le n$.
- For a consistent $A\underline{x} = \underline{b}$, (a) if $\operatorname{nullity} A > 0$, then $A\underline{x} = b$ has infinitely many solns (b) if $\operatorname{nullity} A = 0$, then $A\underline{x} = \underline{b}$ has a unique soln.

Ex 2. $x_1 + x_2 + x_3 = 1$.
$x_1 + 3x_3 = -2+s$
$x_1 - x_2 + r x_3 = 3$.
$r, s \in \mathbb{R}$.

Find r and s such that.
(a) $A\underline{x} = \underline{b}$ is inconsistent.
(b) $A\underline{x} = \underline{b}$ has infinitely many solns.
(c) $A\underline{x} = \underline{b}$ has a unique soln.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & -2+s \\ 1 & -1 & r & 3 \end{bmatrix} \xrightarrow[-r_1 + r_3 \to r_3]{-1r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3+s \\ 0 & -2 & r-1 & 2 \end{bmatrix}$$

$$\xrightarrow{-2r_2 + r_3 \to r_3} \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3+s \\ 0 & 0 & r-5 & 8-2s \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3. & \end{array} \end{array}$$

(a). $r-5=0$, $8-2s \neq 0 \Rightarrow r=5$, $s \neq 4$.
(b) $r-5=0$, $8-2s=0 \Rightarrow r=5$, $s=4$
(c) $r-5 \neq 0$, $8-2s \in \mathbb{R} \Rightarrow r \neq 5$, $s \in \mathbb{R}$.

Thm 1.5 (Test for consistency).

The following are equivalent.

(a) $A\underline{x}=\underline{b}$ is consistent

(b) $\underline{b}$ is a linear combination (l.c.) of the cols of A.

(c) The rref of $[A\ \underline{b}]$ does not have a row $[0\ 0 \cdots 0\ 0\ 1]$.

Pf (a) $\Leftrightarrow$ (c) done above.

Recall $\overset{m\times n}{A} = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_n]$, $\underline{v} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$$A\underline{v} = c_1\underline{a}_1 + c_2\underline{a}_2 + \cdots + c_n\underline{a}_n.$$

$A\underline{x}=\underline{b}$ is consistent iff there is an $n\times 1$ vector $\underline{v}$ such that

$A\underline{v}=\underline{b}$. iff $v_1\underline{a}_1 + v_2\underline{a}_2 + \cdots + v_n\underline{a}_n = \underline{b}$.

iff $\underline{b}$ is a l.c. of cols of A. #.

true/false.

Sec. 1.4. T&F, 10, 16, 34, 42, 47, 92, 75, 77, 79

82, 83, 84.

Find x?

Sec. 1.6. The span of a set of vectors.

Let $S = \{\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}\}$ be a subset of $\mathbb{R}^n$.

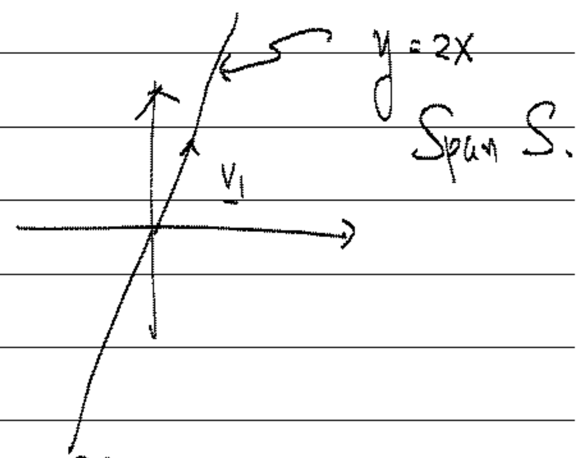
l.c. $c_1\underline{u_1} + c_2\underline{u_2} + \cdots + c_k\underline{u_k}$, where $c_i \in \mathbb{R}$.

Defn. Span S = the set of all l.c's of $\underline{u_1}, \underline{u_2}, \cdots, \underline{u_k}$.

Ex $S = \{\underline{v_1}\}$, where $\underline{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$c_1\underline{v_1}$, where $c_1 \in \mathbb{R}$.

Span $S = \{ c\underline{v_1} \in \mathbb{R}^2 \mid c \in \mathbb{R} \}$

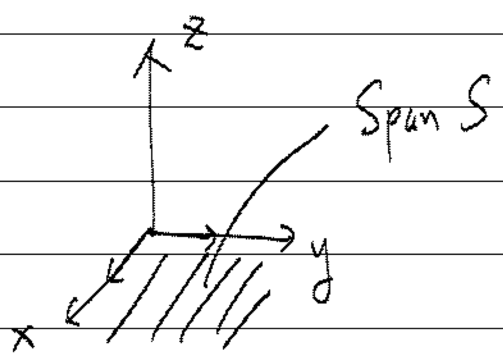


Ex $S = \{\underline{e_1}, \underline{e_2}, \underline{e_3}\}$, $\underline{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Span $S = \mathbb{R}^3$.

$S' = \{\underline{e_1}, \underline{e_2}\}$ $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

Span S' = x-y plane.



Defn If $V = \text{Span } S$, then we say S is a generating set for V OR S generates V.

Claim 1 If S_1 is a subset of S_2, then Span S_1 is also a subset of Span S_2.

Let $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ be a subset of $\mathbb{R}^n$, $\underline{v} \in \mathbb{R}^n$.

Is $\underline{v} \in$ Span S? $\Leftrightarrow$ Is $\underline{v}$ a l.c of $\underline{u}_1, \cdots, \underline{u}_k$?

$\Updownarrow$

Define $n \times k$ $A = [\underline{u}_1 \ \underline{u}_2 \ \cdots \ \underline{u}_k]$

Look at $A\underline{x} = \underline{v}$.

Is $A\underline{x} = \underline{v}$ consistent?

. $\underline{v} \in$ Span S iff $A\underline{x} = \underline{v}$ is consistent.

<u>Ex</u>. $\underline{v} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$, $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}}_{\underline{u}_2}, \underbrace{\begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix}}_{\underline{u}_3} \right\}$.

Is $\underline{v} \in$ Span S? Is $\underline{w} \in$ Span S?

$A = [\underline{u}_1 \ \underline{u}_2 \ \underline{u}_3]$

(i) $[A \ \underline{v}] \xrightarrow{G.E.} [R \ \underline{c}] = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rref.

$A\underline{x} = \underline{v}$ is consistent $\Rightarrow$ $\underline{v} \in$ Span S.

(ii) $[A \ \underline{w}] \xrightarrow{G.E.} [R \ \underline{c}] = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $A\underline{x} = \underline{w}$ is inconsistent $\Rightarrow$ $w \notin$ Span S.

Ex. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$. Show that Span $S = \mathbb{R}^3$

$\underline{u}_1 \quad \underline{u}_2 \quad \underline{u}_3 \quad \underline{u}_4$

- It is clear than Span S is a subset of $\mathbb{R}^3$.

$$A = [\underline{u}_1 \ \underline{u}_2 \ \underline{u}_3 \ \underline{u}_4] \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = R.$$

- For any vector $\underline{b} \in \mathbb{R}^3$, look at $A\underline{x} = \underline{b}$

$$[A \ \ \underline{b}] \xrightarrow{G.E} \left[R \ \begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix} \right]$$ rref. does not have a row $[0\ 0\ 0\ 0\ 1]$.

$A\underline{x} = \underline{b}$ is consistent, $\underline{b} \in$ Span S.

$\Rightarrow \mathbb{R}^3$ is a subset of Span S.

We conclude that Span $S = \mathbb{R}^3$.

- More generally, if the rref of A $(m \times n)$ does not have a zero row (rank $A = m$), then Span of cols of $A = \mathbb{R}^m$.

Thm 1.6. $A: m \times n$. The following are equivalent.

(a) The span of cols of $A = \mathbb{R}^m$.

(b) $A\underline{x} = \underline{b}$ is consistent for every $\underline{b} \in \mathbb{R}^m$.

(c) rank $A = m$

(d) The rref of A does not have a zero row.

Pf. We will prove (b) $\Rightarrow$ (d).

(b) says $A\underline{x} = \underline{b}$ is consistent for every $\underline{b} \in \mathbb{R}^m$.

let R be the rref of A.

$$A \underset{(1)(2)\cdots(p)}{\overset{\text{e.r.o's \#1}}{\rightsquigarrow}} R \underset{(p')\cdots(2')(1')}{\overset{\text{e.r.o's \#2}}{\rightsquigarrow}} A.$$

Now consider $\begin{bmatrix} R & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \end{bmatrix} \overset{\text{e.r.o's \#2}}{\rightsquigarrow} \begin{bmatrix} A & \underline{d} \end{bmatrix}$.

$A\underline{x} = \underline{d}$ is consistent $\Rightarrow$ R does not have a zero row.

(d)

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Thm 1.7. Let $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ be a subset of $\mathbb{R}^n$.

and let $\underline{v} \in \mathbb{R}^n$.

Then $\underbrace{\text{Span}\{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k, \underline{v}\} = \text{Span}\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\}}_{(a)}$ iff $\underbrace{\underline{v} \in \text{Span}\, S}_{(b)}$.

Pf. let $S' = \{\underline{u}_1, \cdots, \underline{u}_k, \underline{v}\}$,

(i) We will first prove (b) $\Rightarrow$ (a).

(b) says $\underline{v} \in \text{Span}\, S$, i.e. $\underline{v} = c_1 \underline{u}_1 + \cdots + c_k \underline{u}_k$ ① for some scalers c_i.

let $\underline{w}$ be any vector in Span S'.

i.e. $\underline{w} = b_1 \underline{u}_1 + \cdots + b_k \underline{u}_k + d\underline{v}$ ② for some scalars b_i, d.

· Substituting ① into ②, we get

$$\underline{w} = (b_1 + c_1 d)\underline{u}_1 + (b_2 + c_2 d)\underline{u}_2 + \cdots + (b_k + c_k d)\underline{u}_k.$$

$\underline{w} \in \text{Span}\, S$.

$\Rightarrow$ Span S' is a subset of Span S.

- From Claim 1, we have Span S is a subset of span S'
we conclude that Span S = Span S'

(ii) Next we will prove (a) $\Rightarrow$ (b). ($\lnot$(b) $\Rightarrow$ $\lnot$(a)).

$\lnot$(b) says $\underline{v} \notin \operatorname{Span} S$.
but $\underline{v} \in \operatorname{Span} S'$

$\Big\}$ $\operatorname{Span} S \neq \operatorname{Span} S'$

$\lnot$(a). #

$\underline{v} = 0 \cdot \underline{u}_1 + \cdots + 0 \cdot \underline{u}_k + 1 \cdot \underline{v}$

Making a generating set smaller:

$$S = \{ \underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k \}.$$

- If some $\underline{u}_i$ is a l.c. of other vectors in S, then $\underline{u}_i$ can be removed without changing its Span.

Sec. 1.6. T&F, 19, 28, 36, 44, 67, 73, 78.

Suppose there are scalars $c_1, c_2, \ldots, c_k$ such that

$$c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_k \underline{u}_k = \underline{0}.$$

Suppose some $c_j \neq 0$, $\underline{u}_j = -\frac{1}{c_j} \left(c_1 \underline{u}_1 + \cdots + c_{j-1} \underline{u}_{j-1} + c_{j+1} \underline{u}_{j+1} + \cdots + c_k \underline{u}_k \right)$

Sec. 1.7. Linear dependence and independence.

<u>Defn</u>. Let $S = \{\underline{u}_1, \cdots, \underline{u}_k\}$ be a subset of $\mathbb{R}^n$.

S is said to be linearly dependent (l. dep) if there are scalars, $c_1, c_2, \cdots, c_k$, not all zero, s.t.

$$c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_k \underline{u}_k = \underline{0}. \qquad (1)$$

If the only set of scalars such that (1) is true is the all zero scalars, then S is linearly independent (l. indep)

Ex. $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$c_1 \underline{e}_1 + c_2 \underline{e}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ iff $c_1 = 0$, $c_2 = 0$.

$\Rightarrow S$ is l. indep.

Ex. $S = \{\underline{0}\}$. $\quad 1 \cdot \underline{0} = \underline{0}$. S is l. dep.

Ex. $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_l, \underline{0}\}$. l. dep.

$$0 \cdot \underline{u}_1 + 0 \cdot \underline{u}_2 + \cdots + 0 \cdot \underline{u}_l + 1 \cdot \underline{0} = \underline{0}.$$

Ex. $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$. $c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k = \underline{0}$.

Define $A = [\underline{u}_1 \ \underline{u}_2 \ \cdots \ \underline{u}_k]$ $\quad \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$.

- Then $c_1\underline{u}_1 + \cdots + c_k\underline{u}_k = \underline{0}$ iff $A\underline{c} = \underline{0}$.

Does $A\underline{x} = \underline{0}$ have a nonzero soln?

- S is l. dep iff $A\underline{x} = \underline{0}$ has a nonzero soln.
- $A\underline{x} = \underline{0}$ is a homogeneous equation and it is always consistent. ($\underline{x} = \underline{0}$ is always a soln).

Ex $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$. $\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$.

$$A\underline{x} = \underline{0} \qquad [A \ \underline{0}] \xrightarrow{G.E.} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{rref.} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & c_i \end{matrix} \end{array}$$

vector form soln:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow S \text{ is l. dep.}$$

- $A\underline{x}=\underline{0}$ has a nonzero soln iff it has a free variable.
  iff nullity $A>0$.
- Cols of A are $\begin{cases} \text{l. dep} & \text{if nullity } A>0 \\ \text{l. indep} & \text{if nullity } A=0. \end{cases}$
- Look at the previous ex, rank $A=3$, nullity $A=4-3>0$
  $\Rightarrow$ cols of $A=S$ is l. dep.

Ex. $A\underline{x}=\underline{0}$ where $A=\begin{bmatrix}1 & -4 & 2 & -1 & 2\\ 2 & -8 & 3 & 2 & 1\end{bmatrix} \to R=\begin{bmatrix}1 & -4 & 0 & 7 & -4\\ 0 & 0 & 1 & -4 & +3\end{bmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

vector form soln

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}+x_2\underbrace{\begin{bmatrix}4\\1\\0\\0\\0\end{bmatrix}}_{\underline{w_1}}+x_4\underbrace{\begin{bmatrix}-7\\0\\4\\1\\0\end{bmatrix}}_{\underline{w_2}}+x_5\underbrace{\begin{bmatrix}4\\0\\-3\\0\\1\end{bmatrix}}_{\underline{w_3}}$$

cols of A are l. dep. $\quad c_1\underline{w_1}+c_2\underline{w_2}+c_3\underline{w_3}=\underline{0}$ iff $c_1=c_2=c_3=0$

Observations:

(1) The vectors $\{\underline{w_1},\underline{w_2},\dots,\underline{w_l}\}$ in vector form soln are always l. indep.

(2) Let A be $m\times n$ with $n>m$. Then nullity $A>0$
Cols of A are l. dep.

Thm 1.8. Let R be the rref of A $(m \times n)$. The following are equivalent.

(a) cols of A are l. indep.

(b) $A\underline{x} = \underline{b}$ has at most one soln for every $\underline{b} \in \mathbb{R}^m$

(c) rank $A = n$ (d) nullity $A = 0$.

(e) Cols of R are distinct standard vectors

(f) The only soln to $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.

(pf) (a) $\Leftrightarrow$ (f). $\Leftrightarrow$ (d). $\Leftrightarrow$ (c) $\underset{?}{\Leftrightarrow}$ (e) (b) ?

We will prove (b) $\Leftrightarrow$ (f).

(i) (b) $\Rightarrow$ (f) (b) says $A\underline{x} = \underline{b}$ has at most one soln for every $\underline{b}$

Take $\underline{b} = \underline{0}$, $A\underline{x} = \underline{b} = \underline{0}$ has at most one soln.

$\Rightarrow$ $A\underline{x} = \underline{0}$ has only one soln $\underline{x} = \underline{0}$.

(f).

(ii) (f) $\Rightarrow$ (b). $(\neg(b) \Rightarrow \neg(f))$.

$\neg$(b) says $A\underline{x} = \underline{b}$ has more than one soln for some $\underline{b}$.

Let $\underline{v_1}, \underline{v_2}$ be two distinct solns of $A\underline{x} = \underline{b}$.

$A\underline{v_1} = \underline{b}$

$A\underline{v_2} = \underline{b}$ and $\underline{v_1} \neq \underline{v_2}$.

$$A\underbrace{(\underline{v_1} - \underline{v_2})}_{\neq \underline{0}} = \underline{b} - \underline{b} = \underline{0}, \Rightarrow A\underline{x} = \underline{0} \text{ has a nonzero soln}$$

(f).

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Summary. $A: m \times n$

Thm 1.6.	$\mathrm{rank}\,A = m$	$A\underline{x} = \underline{b}$ has at least one soln for every $\underline{b} \in \mathbb{R}^m$	cols of A form a generating set for $\mathbb{R}^m$
Thm 1.8	$\mathrm{rank}\,A = n$	$A\underline{x} = \underline{b}$ has at most one soln for every $\underline{b} \in \mathbb{R}^m$	cols of A are l. indep