Sec. 6.3. Orthogonal Projection (OP).

<u>Defn.</u> Orthogonal Complement $S^\perp$ (S perp)

Let S be nonempty subset of $\mathbb{R}^n$.

$S^\perp$ = set of all vectors in $\mathbb{R}^n$ that are orthogonal to each vector in S.

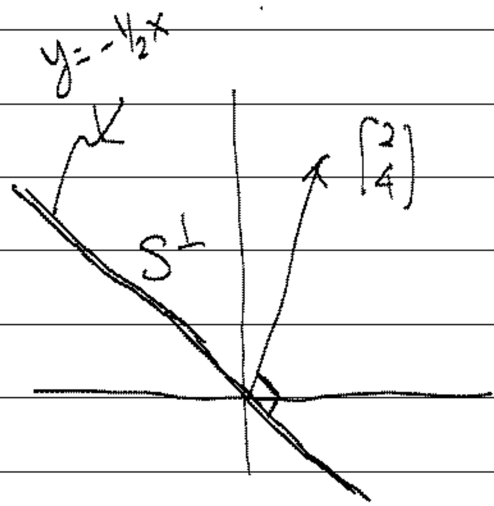
i.e. let $S = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\}$

then $S^\perp = \{\underline{v} \in \mathbb{R}^n \mid \underline{u}_1 \cdot \underline{v} = 0, \underline{u}_2 \cdot \underline{v} = 0, \ldots, \underline{u}_k \cdot \underline{v} = 0\}$

Ex. $S = \mathbb{R}^n$, $S^\perp = \{\underline{0}\}$.

Ex $S = \{\underline{0}\}$, $S^\perp = \mathbb{R}^n$.

Ex. $S = \left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ $S^\perp = ?$



$\underline{v} \in S^\perp$ iff $\underline{v} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$ $\quad 2v_1 + 4v_2 = 0.$

$S^\perp = \{\underline{v} \in \mathbb{Q}^2 \mid v_1 + 2v_2 = 0.\}$

Ex. $S = \{\underline{u}_1, \underline{u}_2\}$. $\quad \underline{v} \cdot \underline{u}_1 = 0$, $\underline{v} \cdot \underline{u}_2 = 0$.

$\underline{u}_1^T \underline{v} = 0$, $\underline{u}_2^T \underline{v} = 0$.

Define $A = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \end{bmatrix}$.

$\underline{v} \in S^\perp$ iff $A\underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

We conclude $S^\perp = \text{Null } A$.

Ex $S = \{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$ $\quad \underline{u}_1^T \underline{v} = 0$, $\underline{u}_2^T \underline{v} = 0$, $\dots$, $\underline{u}_k^T \underline{v} = 0$

Define $A = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_k^T \end{bmatrix}$ $\qquad \underline{v} \in S^\perp$ iff $A\underline{v} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$S^\perp = \text{Null } A.$$

Properties of $S^\perp$

(1) $S^\perp$ is a subspace of $\mathbb{R}^n$.

(2) $S^\perp = (\text{Span } S)^\perp$ (Problem 57).

(3) If V is a subspace of $\mathbb{R}^n$, then $(V^\perp)^\perp = V$ (Prob 60)

(4) If $\underline{v} \in S$ and $\underline{v} \in S^{\perp}$, then $\underline{v} = \underline{0}$.

$S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ $\quad A = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_k^T \end{bmatrix}$

$\text{Row } A = \text{Span}\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k\} = \text{Span } S = \text{Col } A^T.$

$S^{\perp} = (\text{Span } S)^{\perp} = \text{Null } A.$

We conclude that $\text{Null } A = (\text{Row } A)^{\perp} = (\text{Col } A^T)^{\perp}$

$$\underbrace{\begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_k^T \end{bmatrix}}_{A.} \underline{v} = \underline{0}.$$

Thm 6.7 Orthogonal Decomposition Thm

Let W be subspace of $\mathbb{R}^n$. Then for every vector $\underline{v} \in \mathbb{R}^n$, there exist a unique $\underline{w} \in W$, and a unique $\underline{z} \in W^{\perp}$ such that

$$\underline{v} = \underline{w} + \underline{z}.$$

In addition, if $\{\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_k\}$ is an orthonormal basis for W, then

$$\underline{w} = (\underline{v} \cdot \underline{w}_1)\underline{w}_1 + (\underline{v} \cdot \underline{w}_2)\underline{w}_2 + \cdots + (\underline{v} \cdot \underline{w}_k)\underline{w}_k \quad (1)$$

Pf Let $\mathcal{B} = \{\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_k\}$ be orthonormal basis for W.

and let $\underline{w}$ be the vector defined in (1).

Clearly $\underline{w} \in W$.

Define $\underline{z} = \underline{v} - \underline{w}$.

Next we will prove that $\underline{z} \in W^{\perp}$.

From (1), $\underline{z} \cdot \underline{w}_1 = (\underline{v} - \underline{w}) \cdot \underline{w}_1$

$$= \underline{v} \cdot \underline{w}_1 - \Big((\underline{v} \cdot \underline{w}_1) \overbrace{\underline{w}_1 \cdot \underline{w}_1}^{=1} + (\underline{v} \cdot \underline{w}_2) \overbrace{\cancel{\underline{w}_2 \cdot \underline{w}_1}}^{0} + \cdots + (\underline{v} \cdot \underline{w}_k) \overbrace{\cancel{\underline{w}_k \cdot \underline{w}_1}}^{0} \Big)$$

$$= 0$$

$\underline{z} \cdot \underline{w}_2 = \underline{v} \cdot \underline{w}_2 - \left((\underline{v} \cdot \underline{w}_1) \cancel{\underline{w}_1 \cdot \underline{w}_2} + (\underline{v} \cdot \underline{w}_2) \underline{w}_2 \cdot \underline{w}_2 + \cdots + (\underline{v} \cdot \underline{w}_k) \cancel{\underline{w}_k \cdot \underline{w}_2} \right)$

$= 0$

$\ldots \quad \underline{z} \cdot \underline{w}_k = 0.$

$$\underline{z} \in B^{\perp} = (\mathrm{Span}\, B)^{\perp} = W^{\perp}.$$

$$\underline{v} = \underset{\in W}{\underline{w}} + \underset{\in W^{\perp}}{\underline{z}}.$$

Uniqueness. Suppose $\underline{v} = \underline{w}' + \underline{z}'$, $\underline{w}' \in W$ and $\underline{z}' \in W^{\perp}$

$$\underline{v} = \underline{w} + \underline{z} = \underline{w}' + \underline{z}'$$

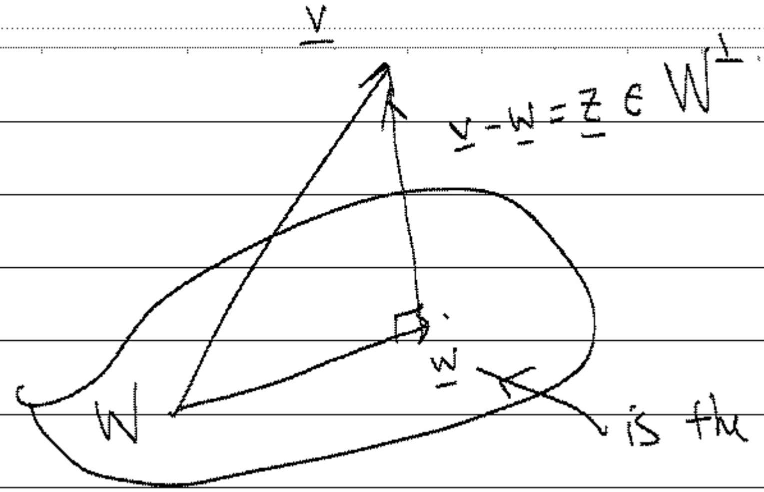
$$\underset{\in W}{\underline{w} - \underline{w}'} = \underset{\in W^{\perp}}{\underline{z}' - \underline{z}}.$$

We can conclude from Property 4 that $\underline{w} - \underline{w}' = \underline{0}$ and $\underline{z}' - \underline{z} = \underline{0}$

$\Rightarrow \underline{w}' = \underline{w}, \quad \underline{z}' = \underline{z}.$ #

Defn Orthogonal Projection onto W. (OP)

Let W be a subspace of $\mathbb{R}^n$. The OP of $\underline{v}$ onto W is the unique vector $\underline{w} \in W$ such that $(\underline{v} - \underline{w})$ is in $W^{\perp}$.



is the OP of $\underline{v}$ onto W.

What is $\underline{z}$? $\underline{z}$ is the OP of $\underline{v}$ onto $W^{\perp}$.

$$\underline{v} = \underset{\in W^{\perp}}{\underline{z}} + \underset{\in (W^{\perp})^{\perp}}{\underline{w}}.$$

$$\underline{v} = \underline{w} + \underline{z}$$

OP of $\underline{v}$ onto $W^{\perp}$.

OP of $\underline{v}$ onto W

Claim 1 Let $\mathcal{B}_1$ be a basis for W.

Let $\mathcal{B}_2$ be a basis for $W^\perp$.

Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $\mathbb{R}^n$.

Proof ① $\text{Span}(\mathcal{B}_1 \cup \mathcal{B}_2) = \mathbb{R}^n$ by Thm 6.7.

② $\mathcal{B}_1 \cup \mathcal{B}_2$ is l. indep. (one of the problem).

$$\dim W + \dim W^\perp = n.$$

Ex 1 Let $W = \text{Null } A$, $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$, and $\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Find the OP of $\underline{v}$ onto W.

a basis $\xrightarrow[\text{Schmidt}]{\text{Gram-}}$ orthogonal basis $\xrightarrow{\text{normalization}}$ orthonormal basis.

$$A\underline{x} = \underline{0} \longrightarrow \Big\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\underline{u}_1}, \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{u}_2} \Big\} \text{ a basis for } W.$$

Gram - Schmidt $\{\underline{v}_1, \underline{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ orthogonal basis for W.

$\{\underline{w}_1, \underline{w}_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ orthonormal basis.

$\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, from (1) in Thm 6.7, $\underline{w} = \underbrace{(\underline{v} \cdot \underline{w}_1)}_{= \frac{4}{\sqrt{2}}} \underline{w}_1 + \underbrace{(\underline{v} \cdot \underline{w}_2)}_{= \frac{6}{\sqrt{3}}} \underline{w}_2$.

$$\underline{w} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}.$$

$$\underline{z} = \underline{v} - \underline{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad (\underline{z} \cdot \underline{w} = 0)$$

Define $U_W : \mathbb{R}^n \to \mathbb{R}^n$ $\quad U_W(\underline{v}) = \underline{w}$, OP of $\underline{v}$ onto W

W subspace.

- U_W is called the orthogonal projection operator.

and U_W is linear.

Defn. the standard matrix of U_W, P_W $(n \times n)$ is called the OP matrix of W.

Thm 6.8 Let C be an $n \times k$ matrix whose columns form a basis for W. Then

$$P_W = C\left(C^T C\right)^{-1} C^T.$$

$\operatorname{rank} C = k$, $C = \left[\underline{u}_1\ \underline{u}_2\ \cdots\ \underline{u}_k\right]_{n \times k}$. $W = \operatorname{Span}\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$.

Claim 2 $\overset{k\times k}{C^T C}$ is invertible.

Pf. Suppose $\underline{d}$ $(k\times 1)$ such that $C^T C\,\underline{d} = \underline{0}$.

Look at $\underline{d}^T (C^T C)\,\underline{d} = 0$.

$$\|C\underline{d}\|^2 = (C\underline{d})^T (C\underline{d}) = \underline{d}^T (C^T C)\,\underline{d} = 0.$$

$$\Rightarrow C\underline{d} = \underline{0} \Rightarrow \underline{d} = \underline{0} \quad (\because \underline{u}_1, \dots, \underline{u}_k \text{ are l. indep})$$

$\Rightarrow$ The only soln to $C^T C\,\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$ $\Rightarrow$ $C^T C$ is invertible. #

Proof of Thm 6.8.

let $W = \text{Span}\{\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}\}$ and $C = [\underline{u_1}\ \underline{u_2} \cdots \underline{u_k}]$

and $\text{rank}\, C = k$.

For any $\underline{v} \in \mathbb{R}^n$, let $\underline{w} = U_W(\underline{v})$.

From Thm 6.7, $\underline{z} = \underline{v} - \underline{w} \in W^\perp$.

Because $\underline{w} \in W$, $\underline{w} = C\underline{u}$ for some $\underline{u}$.

$\underline{z} = \underline{v} - \underline{w} \in W^\perp \Rightarrow \underline{z}$ is orthogonal to $\underline{u_1}, \ldots, \underline{u_k}$

$C^T \underline{z} = \underline{0}$. $\quad C^T = \begin{bmatrix} \underline{u_1}^T \\ \vdots \\ \underline{u_k}^T \end{bmatrix}$

$C^T(\underline{v} - \underline{w}) = \underline{0}$

$C^T(\underline{v} - C\underline{u}) = \underline{0}$.

$C^T C\, \underline{u} = C^T \underline{v}$

$\underline{u} = (C^T C)^{-1} C^T \underline{v}$.

$\underline{w} = U_W(\underline{v}) = C\underline{u} = C(C^T C)^{-1} C^T \underline{v}$ for any $\underline{v} \in \mathbb{R}^n$.

$C(C^TC)^{-1}C^T$ is the standard matrix for O_W.

" P_W OP matrix #

Ex 1 (conti) Find OP of $\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ onto $W = \text{Span}\,[1\ {-1}\ 2]$.

from previous ex, we have $\{\underline{u}_1, \underline{u}_2\}$ a basis for W.

Form $C = [\underline{u}_1\ \underline{u}_2] = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$P_W = C(C^TC)^{-1}C^T = \frac{1}{6}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

$$P_W \underline{v} = P_W \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \underline{w},$$ OP of $\underline{v}$ onto W

Same as previous ex.

If we choose $C = [\underline{w}_1\ \underline{w}_2]$ ($\{\underline{w}_1, \underline{w}_2\}$ orthonormal basis for W)

then $C^TC = I_2$. OP matrix P_W

$\begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \end{bmatrix} [\underline{w}_1\ \underline{w}_2]$ becomes $P_W = CC^T$.

Thm 6.7

$$\underline{v} = \underline{w} + \underline{z} \qquad \dim W + \dim W^{\perp} = n$$

$\underline{w}$: OP of $\underline{v}$ onto W, $\underline{z}$: OP of $\underline{v}$ onto $W^{\perp}$.

$W = \text{Null } A, \quad A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$A\underline{x} = \underline{0}$.

$W^{\perp} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{= \underline{y}} \right\}$.

OP of $\underline{v}$ onto $W^{\perp}$, $\underline{z} = \dfrac{\underline{v} \cdot \underline{y}}{\|\underline{y}\|^2} \underline{y} = \dfrac{6}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

OP of $\underline{v}$ onto $W = \underline{v} - \underline{z} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \qquad = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Claim $P_W + P_{W^{\perp}} = I$.

For any $\underline{v} \in \mathbb{R}^n$, $(P_W + P_{W^{\perp}})\underline{v} = P_W \underline{v} + P_{W^{\perp}} \underline{v}$

$= \underline{w} + \underline{z} = \underline{v}$.

$W = \text{Null } A, \quad A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}.$

$W^{\perp} = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\underline{y}} \right\}. \qquad C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

OP matrix of $W^{\perp}$

$$P_{W^{\perp}} = C\left(C^TC\right)^{-1}C^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} (6)^{-1} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$$P_W = I_3 - P_{W^{\perp}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$ same as previous ex.