Ch 5 Eigenvalues and Eigenvectors

<u>Defn</u> A L.T. with the same domain and codomain, then we say it is a linear operator.

$T: \mathbb{R}^n \to \mathbb{R}^n$ a linear operator on $\mathbb{R}^n$.

- its standard matrix A will be $n \times n$.

<u>Defn</u> Let T be a linear operator on $\mathbb{R}^n$.
A nonzero vector $\underline{v}$ of $\mathbb{R}^n$ is said to be an eigenvector of T if $T(\underline{v}) = \lambda \underline{v}$ for some scalar λ.
The scalar λ is called an eigenvalue of T.

<u>Defn</u> a nonzero vector $\underline{v}$ of $\mathbb{R}^n$ is said to be an eigenvector of A $(n \times n)$ if $A\underline{v} = \lambda \underline{v}$ for some scalar λ.

λ: eigenvalue.

Let A be the standard matrix of T.

$T(\underline{v}) = A\underline{v}$.

- T and its standard matrix A have the same eigenvalues and eigenvectors.
- ~~all matrices in Ch 5 are square matrices.~~

- Given an eigenvector $\underline{v}$ of A, how to find its eigenvalue?

$$A\underline{v} = \lambda \underline{v}.$$

- Given an eigenvalue λ of A, how to find its eigenvectors?

$$A\underline{v} = \lambda\underline{v} \Rightarrow (A - \lambda I_n)\underline{v} = \underline{0}.$$

- eigenvectors are the nonzero soln to $(A - \lambda I_n)\underline{x} = \underline{0}$.

$$\text{Null}\ (A - \lambda I_n).$$

- Defn Eigenspace of A corresponding to eigenvalue λ

$$E_\lambda = \text{Null}\ (A - \lambda I_n).$$

- eigenspace E_λ is a subspace of $\mathbb{R}^n$ and all its nonzero vectors are eigenvectors of A.

if $\underline{v} \in E_\lambda$, then $A\underline{v} = \lambda\underline{v}$.

Ex. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1+x_2 \end{bmatrix}$. $\lambda_1 = 3$ and $\lambda_2 = -2$ are eigenvalues of T. Find a basis for each eigenspace.

$A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$.

$\underline{\underline{E_{\lambda_1}}}$ $(A - \lambda_1 I_2)\underline{x} = \underline{0}$. $A - 3I_2 \xrightarrow{G.E.} \begin{bmatrix} 1 & 2/3 \\ 0 & 0 \end{bmatrix}$ rref.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}.$$

$\left\{ \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{λ_1}.

$\underline{\underline{E_{\lambda_2}}}$ $(A - \lambda_2 I_2) \xrightarrow{G.E.}$ rref. exercise.

Ex. $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. $\lambda = 3$ is an eigenvalue of B. Find a basis for E_λ.

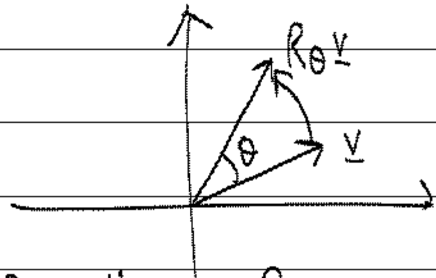
$B - 3I \xrightarrow{G.E.} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rref.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_λ.

$\underline{\text{Ex}}$ rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$\underline{v} \in \mathbb{R}^2$

$R_\theta \underline{v}$



- if $\theta \neq 0^\circ$ or 180°, then $R_\theta \underline{v}$ is not parallel to $\underline{v}$ for any nonzero vector $\underline{v}$; i.e. $A\underline{v} \neq \lambda \underline{v}$ for any nonzero $\underline{v}$.

- if $\theta \neq 0^\circ$ or 180°, then R_θ does not have any eigenvector and eigenvalues.

Sec. 5.2. Characteristic Polynomials. C.P.

λ is an eigenvalue of A iff $A\underline{v} = \lambda \underline{v}$ for some nonzero $\underline{v}$.

iff $(A - \lambda I)\underline{X} = \underline{0}$ has nonzero solns.

iff $\det(A - \lambda I) = 0$. $\quad A: n \times n$.

$\underline{\text{Defn}}$ C.P. of $A = \det(A - tI_n)$.

- CP of A is a polynomial in t.

Ex. $A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ Find CP of A.

$$\det(A - tI_2) = \det \begin{bmatrix} 0-t & -2 \\ -3 & 1-t \end{bmatrix} = t^2 - t - 6 = (t-3)(t+2)$$

- eigenvalues of A are the roots of the CP of A.

$\lambda_1 = 3$, $\lambda_2 = -2$ are eigenvalues of $\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$.

Ex. $A = \begin{bmatrix} -4 & -5 \\ 5 & 6 \end{bmatrix}$ $\det(A - tI) = \det \begin{bmatrix} -4-t & -5 \\ 5 & 6-t \end{bmatrix}$,

$$= t^2 - 2t + 1 = (t-1)^2.$$

$\lambda = 1$ is an eigenvalue of A.

<u>Ex</u>. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ CP of $A = \det \begin{bmatrix} 0-t & -1 \\ 1 & 0-t \end{bmatrix} = t^2 + 1$.

A has no eigenvalue.

A : $n \times n$

$$\text{CP of } A = \det \begin{bmatrix} a_{11}-t & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-t & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & & \cdots & a_{nn}-t \end{bmatrix}$$

$$\| \quad \det(A-tI)$$

$(n-2)$ t's.

$$= (a_{11}-t)\det\begin{bmatrix} a_{22}-t & \cdots & \\ \vdots & \ddots & \\ & & a_{nn}-t \end{bmatrix} + (-1)\,a_{12}\det\begin{bmatrix} \cdots \end{bmatrix} + a_{13}\det\begin{bmatrix} \ \end{bmatrix} + \cdots + (-1)^{1+n} a_{1n}\det\left(\ \right)$$

$$= (a_{11}-t)(a_{22}-t)\det\begin{bmatrix} a_{33}-t & \cdots \\ & \ddots \\ \cdots & a_{nn}-t \end{bmatrix} + \underbrace{\quad}_{\text{at most } (n-2)\ t\text{'s}}$$

$$= (a_{11}-t)(a_{22}-t)\cdots(a_{nn}-t) + \cdots \text{(at most } (n-2)\ t\text{'s)}.$$

$$= (-1)^n t^n + \underbrace{p_{n-1}}_{(-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})} t^{n-1} + p_{n-2} t^{n-2} + \cdots + p_1 t^1 + p_0$$

- The CP of an $n \times n$ matrix is an nth degree polynomial in t.
- $p_0 = \text{CP of } A \big|_{t=0} = \det A.$

Defn. Multiplicity of eigenvalue λ.

e.g. CP of $A = (-1)^{15} (t-1)^3 (t+2)^5 (t-6)^7$.

$\lambda_1 = 1, \quad \lambda_2 = -2, \quad \lambda_3 = 6.$

$m_1 = 3, \quad m_2 = 5, \quad m_3 = 7$

- multiplicity of λ is the largest integer k such that $(t-\lambda)^k$ is a factor of $\det(A - tI_n)$.

Ex. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. $\det[A - tI] = \det \begin{bmatrix} -1-t & 0 & 0 \\ 0 & 1-t & 2 \\ 0 & 2 & 1-t \end{bmatrix} = -(t+1)^2(t-3)$

$\lambda_1 = -1, \quad \lambda_2 = 3$

$m_1 = 2, \quad m_2 = 1.$

$\underline{E_{\lambda_1}}$ $(A - (-1)I_3) \xrightarrow{G.E.} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rref.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_{λ_1}.

$\dim E_{\lambda_1} = 2 = m_1.$

$\underline{E_{\lambda_2}}$ $A-(3)I_3 \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. rref.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } E_{\lambda_2}.$$

$\dim E_{\lambda_2} = 1 = m_2.$

Ex. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix}$. $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

$$\text{CP of } A = \det \begin{bmatrix} -1-t & 0 & 0 \\ 2 & -1-t & -1 \\ 0 & 0 & -1-t \end{bmatrix} = (-1)^3 (t+1)^3 \leftarrow \text{CP of } -I_3$$

$\lambda_1 = -1 \quad m_1 = 3.$

$\underline{E_{\lambda_1}}$ $\left(A-(-1)I_3\right) \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ rref.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } E_{\lambda_1}$$

$\dim E_{\lambda_1} = 2 \neq m_1.$

Thm 5.1. $1 \leq \dim E_\lambda \leq$ multiplicity of λ.

Similar Matrices. $A, B : n \times n$

- A is similar to B if $B = P^{-1}AP$ for some invertible P.
- If A is similar to B, then B is also similar to A.

We say A and B are similar.

<u>Claim</u> Similar matrices have the same CP.

<u>Pf.</u> CP of $B = \det(B - tI) = \det(P^{-1}AP - tI)$

$$= \det P \det(P^{-1}AP - tI) \det P^{-1}$$

$$= \det\left(P(P^{-1}AP - tI)P^{-1}\right) = \det(A - tI)$$

$$= \text{CP of } A.$$

- In particular, it implies that similar matrices have the same eigenvalues with the same multiplicity.

- In P84, if $\{\underline{v_1}, \dots, \underline{v_k}\}$ is a basis for E_λ of A.

Then $\{P\underline{v_1}, P\underline{v_2}, \dots, P\underline{v_k}\}$ is a basis for E_λ of B.

$$B = P^{-1}AP.$$

dim of E_λ of A = dim of E_λ of B.

Proof of $\dim E_\lambda \leq$ multiplicity of λ.

$V^{-1}V = I_n$

Let $V = [\underline{v_1} \ \underline{v_2} \ \cdots \ \underline{v_n}]$ be $n \times n$ invertible.

Then $V^{-1}\underline{v_i} = \underline{e_i}$.

Let A be $n \times n$ and λ be an eigenvalue of A.

Let $\dim E_\lambda = k$ and let $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_k}\}$ be a basis for E_λ.

i.e. $A\underline{v_1} = \lambda \underline{v_1}$, $A\underline{v_2} = \lambda \underline{v_2}$ $\cdots$ $A\underline{v_k} = \lambda \underline{v_k}$

- We can extend $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_k}\}$ so that $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_k}, \underline{v_{k+1}}, \dots, \underline{v_n}\}$ becomes a basis for $\mathbb{R}^n$.
- define $V = [\underline{v_1} \ \underline{v_2} \ \cdots \ \underline{v_n}]$ $n \times n$ invertible.

Calculate $V^{-1}AV = V^{-1}A\,[\underline{v}_1\ \underline{v}_2 \cdots \underline{v}_k\ \underline{v}_{k+1} \cdots \underline{v}_n]$.

$$= V^{-1}\,[\lambda\underline{v}_1\ \lambda\underline{v}_2 \cdots \lambda\underline{v}_k\ A\underline{v}_{k+1} \cdots A\underline{v}_n].$$

$$= [\lambda\underline{e}_1\ \lambda\underline{e}_2 \cdots \lambda\underline{e}_k\ \underline{u}_{k+1} \cdots \underline{u}_n].$$

$k\times(n-k)$

$$= \underbrace{\begin{bmatrix} \lambda & 0 & & 0 & * & * & * & * \\ 0 & \lambda & & \vdots & & \vdots & & \\ \vdots & 0 & \ddots & \lambda & * & * & \cdots & * \\ \vdots & \vdots & & 0 & & & & \\ \vdots & \vdots & & \vdots & * & * & \cdots & * \\ & & & & * & * & \cdots & \\ 0 & 0 & & 0 & * & * & \cdots & \end{bmatrix}}_{k\text{ cols.}} = \begin{bmatrix} \lambda I_k & B \\ O & C \end{bmatrix}$$

$(n-k)\times k$. $(n-k)\times(n-k)$.

CP of A = CP of $V^{-1}AV$. ($\because$ A & $V^{-1}AV$ are similar)

$$= \det(V^{-1}AV - tI_n) = \det\begin{bmatrix} (\lambda-t)I_k & B \\ O & C - tI_{n-k} \end{bmatrix}$$

$$= \det((\lambda-t)I_k)\ \det(C - tI_{n-k}).$$

$$= (\lambda-t)^k \det(C - tI_{n-k}).$$ this polynomial may or may not contain the factor $(t-\lambda)$.

$\Rightarrow$ multiplicity of $\lambda \geq k = \dim E_\lambda$. #

Exercise

Sec. 5.1 T&F, 24, 40, 61, 62, 66, 67, 69, 72.

Sec. 5.2. T&F, 10, 21, 24, 40, 74, 75, 83 ~ 86.