Ch 1. Matrices, Vectors and Systems of linear equations

Matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \longleftarrow \text{row}$$

column (col.)

3 rows 2 columns.

3×2, 3 by 2.

- The entry at the i^{th} row and j^{th} col. is denoted by a_{ij}

  $a_{12} = 2$, $a_{31} = 5$

- a scalar is a real number.
- the set of all real number $\mathbb{R}$.
- square matrix 3×3, $n \times n$, $m \times m$.
- Matrix addition $A, B : m \times n$.

  $C = A + B$ $c_{ij} = a_{ij} + b_{ij}$.

- Scalar Multiplication.

  $D = cA$ $d_{ij} = c\, a_{ij}$, c : scalar.

- Zero matrix $\mathbb{O} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$.

Thm 1.1 Properties of Addition and Scalar Multiplication
(see text book)

- Transpose. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}$.

(i,j) th entry of A^T = (j,i) th entry of A.

A: $m \times n$ A^T: $n \times m$.

- If $A^T = A$, then A is called a symmetric matrix.
If $A^T = -A$, then A is called a skew-symmetric matrix (anti).

- Prob 81, For any $n \times n$ matrix A, there is a unique symmetric B and a unique skew-symmetric C such that
$A = B + C$.

Thm 1.2. A, B : $m \times n$, $s \in \mathbb{R}$.

(a) $(sA)^T = s \cdot A^T$

(b) $(A+B)^T = A^T + B^T$

(c) $(A^T)^T = A$.

Vectors

- In textbook / exams, lower-case boldfaced letters.

  $\mathbf{u}, \mathbf{v}, \mathbf{w}$

  In lectures, $\underline{u}, \underline{v}, \underline{w}$.

- $n \times 1$ vector $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\quad 1 \times n$ $\underline{u} = [u_1 \ u_2 \ \cdots \ u_n]$.

- The set of all $n \times 1$ or $1 \times n$ vectors is denoted by $\mathbb{R}^n$

- If $\underline{w} \in \mathbb{R}^n$, then $\underline{w}$ is an $n \times 1$ or $1 \times n$ vector.

Sec. 1.2 Linear Combinations (l.c.)

Defn. Let $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k \in \mathbb{R}^n$.

A l.c. of $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k$ is a vector of the form:

$$c_1 \underline{u}_1 + c_2 \underline{u}_2 + \cdots + c_k \underline{u}_k, \quad \text{where } c_1, c_2, \cdots, c_k \in \mathbb{R}$$

$n \times 1$.

Ex. $\underline{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

$$\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\underline{u}_1 + (-1)\underline{u}_2.$$

$\underline{v}$ is a l.c. of $\underline{u}_1$ and $\underline{u}_2$.

- Standard Vectors.

2×1 $\quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

3×1 $\quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

$n \times 1$ $\quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

- $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots + v_n \underline{e}_n.$

- every $n \times 1$ vector can be written as a l.c. of the standard vectors of $\mathbb{R}^n$.

Defn Identity Matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 \end{bmatrix}.$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix}.$$

$$I_n = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_n \end{bmatrix}.$$

Matrix - Vector Multiplication.

Let $m \times n$ $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $\quad n \times 1$ $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix}$$

Defn

$$\overset{m \times n}{A} \; \overset{n \times 1}{\underline{v}} = v_1 \underline{a}_1 + v_2 \underline{a}_2 + \cdots + v_n \underline{a}_n \quad m \times 1.$$

- $A\underline{v}$ is a l.c. of cols of A.

$$A\underline{v} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Ex. $A\underline{e}_1 = 1 \cdot \underline{a}_1 + 0 \cdot \underline{a}_2 + \cdots + 0 \cdot \underline{a}_n = \underline{a}_1$.

$A\underline{e}_2 = 0 \cdot \underline{a}_1 + 1 \cdot \underline{a}_2 + 0 \cdot \cdots + 0 \cdot \underline{a}_n = \underline{a}_2$.

$A\underline{e}_j = 0 \cdot \underline{a}_1 + \cdots + 0\,\underline{a}_{j-1} + 1 \cdot \underline{a}_j + 0 \cdots + 0 \cdot \underline{a}_n = \underline{a}_j$

- $A\underline{e}_j = \underline{a}_j$, the j^{th} col of A.

Thm 1.3 see textbook.

(d) $A\underline{e}_j = \underline{a}_j$.

(e) Given 2 $m \times n$ matrices A, B, if $A\underline{w} = B\underline{w}$ for every $\underline{w} \in \mathbb{R}^n$, then $A = B$.

Pf. $A\underline{w} = B\underline{w}$ for all $\underline{w} \in \mathbb{R}^n$.

choose $\underline{w} = \underline{e}_1$, then $A\underline{e}_1 = B\underline{e}_1 \Rightarrow \underline{a}_1 = \underline{b}_1$

" $\underline{w} = \underline{e}_2$, — — — — — $\underline{a}_2 = \underline{b}_2$

$\vdots$

$\underline{w} = \underline{e}_n$, — — — — — $\underline{a}_n = \underline{b}_n$

$\left.\begin{matrix} \underline{a}_1 = \underline{b}_1 \\ \underline{a}_2 = \underline{b}_2 \\ \vdots \\ \underline{a}_n = \underline{b}_n \end{matrix}\right\} A = B.$

Sec. 1.3 Systems of Linear Equations.

unknowns / variables : $x_1, x_2, \cdots, x_n$.

scalars $a_1, a_2, \cdots, a_n, b$.

(l. eqn)

Defn A linear equation in $x_1, x_2, \cdots, x_n$ is an equation of the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$a_1, a_2, \cdots, a_n$: coefficients.

System of l. eqns.

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m \end{aligned}$$

m eqns, n unknowns / variables.

Define coefficient matrix.

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & & x_n \end{matrix} \end{array} m \times n, \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} m \times 1, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} n \times 1.$$

Then the system of l eqns can be written as.

$$\underset{m \times n}{A} \ \underset{n \times 1}{\underline{x}} = \underset{m \times 1}{\underline{b}}.$$

Defn. The $m \times (n+1)$ matrix $[A \ \ \underline{b}]$ is called the augmented matrix of the system of l. eqns $A\underline{x} = \underline{b}$.

Defn The set of all solutions to $A\underline{x} = \underline{b}$ is called the solution set of $A\underline{x} = \underline{b}$.
(soln)

Defn If two systems of l. eqns have the same soln. set, then they are equivalent.

Defn. A system of l. eqns is said to be consistent if it has a soln. or solutions.

Defn. - - - - - - - - - - - inconsistent if it has no soln.

Elementary Row Operations (e.r.o.)

$m \times n$ $A = \begin{bmatrix} \underline{r_1} \\ \underline{r_2} \\ \vdots \\ \underline{r_m} \end{bmatrix}$ $\qquad \underline{r_i}$: ith row of A.

(i) interchange row i and row j. $\underline{r}_i \leftrightarrow \underline{r}_j$

(ii) multiply row i by a nonzero scalar $c \cdot \underline{r}_i \ (c \neq 0)$.

(iii) add a multiple of row i to row j. $c\underline{r}_i + \underline{r}_j \to \underline{r}_j$

Ex, $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 3 & -6 & -5 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \end{bmatrix}$

$A \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 2 & -1 & 1 & 0 \\ 3 & -6 & -5 & 1 \\ 1 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} A.$

$A \xrightarrow{-2 r_2} \begin{bmatrix} 1 & -1 & 2 & -1 \\ -6 & 12 & 10 & -2 \\ 2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{-1/2\, r_2} A.$

$A \xrightarrow{-1 \cdot r_1 + r_3 \to r_3} \begin{bmatrix} 1 & -1 & 2 & -1 \\ 3 & -6 & -5 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{1 \cdot r_1 + r_3 \to r_3} A.$

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① In Prob 83, e.r.o's are reversible by the e.r.o's of the same type.

② In Ch 2, if $[A \ \underline{b}] \xrightarrow{e.r.o} [A' \ \underline{b}']$, then $A\underline{x} = \underline{b}$ and $A'\underline{x} = \underline{b}'$ are equivalent.

Defn. Zero row $[0\ 0\ \cdots\ 0]$.

Non zero row $[0\ 0\ 0\ -3\ 5\ 7\ 0\ 0\ 0\ -2]$.

leading entry.

Defn. Row Echelon Form (r.e.f.).

A matrix is said to be in r.e.f. if it satisfies.

(i) Each nonzero row lies above each zero row.

(ii) The leading entry at the bottom row is at the right hand side of the leading entry at the top row.

$$\begin{bmatrix} 0 & 0 & 5 & 7 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -3 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(iii) The entries below leading entries are zero.

- In fact, (i) + (ii) will imply (iii).

Defn. Reduced Row Echelon Form (rref)

A matrix is said to be in rref if it satisfies

(i) (ii) (iii) and

(iv) the leading entries are $\underline{1}$.

(v) the entries above the leading entries are zero.

Ex. $\begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

rref.

Ex. $x_1 - 3x_2 + 2x_4 = 7$

$x_3 + 6x_4 = 9$

$x_5 = 2$

$0 = 0$

$x_1 = 7 + 3x_2 - 2x_4.$

$x_3 = 9 - 6x_4.$

$x_5 = 2$

rref. $0 = 0$.

$$[A \ \underline{b}] = \begin{array}{c} \begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & b_i \end{matrix} \end{array}$$

$x_2 = 0$
$x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 9 \\ 0 \\ 2 \end{bmatrix}$$

$X_2 = 1$

$X_4 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

- The general soln to $A\underline{x} = \underline{b}$ in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 9 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}$$

Ex. $A\underline{x} = \underline{b}$ with

$$\begin{bmatrix} A & \underline{b} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & b_i \end{array} \end{array} \text{ rref}$$

$x_1 - 3x_3 = 0$

$x_2 + 2x_3 = 0$

$0 = 1$

$0 = 0$

No soln!

- If $[A \ \underline{b}]$ has a row of the form $[0 \ 0 \cdots 0 \ 0 \ \overset{c \neq 0}{c}]$, then $A\underline{x} = \underline{b}$ is inconsistent.

$0 = c$

Thm 1.4 Every matrix can be transformed into one and only one r.r.e.f. by a sequence of e.r.o's. (proof in Ch 2 and Appendix E).

Sec. 1.1. 75, 79, 81, 82,

Sec. 1.2. 71, 74, 79, 81.

Sec. 1.3. 48, 51, 53, 81, 83.

pivot positions   pivot positions

r.r.e.f.

$$[A\ \underline{b}] = \begin{array}{cccccc} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right] \\ \underline{a}_1\ \ \underline{a}_2\ \ \underline{a}_3\ \ \underline{a}_4\ \ \underline{a}_5\ \ \underline{b} \end{array} \sim \begin{array}{c} \left[\begin{array}{cccccc} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \underline{r}_1\ \ \underline{r}_2\ \ \underline{r}_3\ \ \underline{r}_4\ \ \underline{r}_5\ \ \underline{c} \end{array} = [R\ \underline{c}].$$

pivot cols.   pivot cols.

$\underline{a}_2$, $\underline{a}_5$: non pivot cols. $\underline{r}_2$, $\underline{r}_5$: non pivot cols.

<u>Defn</u>. basic variables: varibles that correspond to pivot cols. (x_1, x_3, x_4).

free variables: varibles that correspond to non pivot cols. (x_2, x_5)

Sec. 1.4 Gaussian Elimination (G.E.)

$$[A \quad \underset{\sim}{b}] = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 6 & 3 & 9 \end{bmatrix} \xrightarrow[\substack{-2r_1+r_3\to r_3 \\ 3r_1+r_4\to r_4}]{1r_1+r_2\to r_2} \begin{bmatrix} 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix} \xrightarrow{-1r_2+r_4\to r_4} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{bmatrix}$$

$$\xrightarrow{-2r_3+r_4\to r_4} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{ref.}$$

↑ forward pass of G.E.

↓ backward pass

$$\xrightarrow{1/4 \times r_3} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-2r_3+r_1\to r_1]{2r_3+r_2\to r_2} \begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-1\, r_2} \begin{bmatrix} \cdots \end{bmatrix} \xrightarrow{r_2+r_1\to r_1} \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & c_i \end{matrix} \end{array} = [R \; c]$$

$[R\ \underset{\sim}{c}]$ is the rref of $[A\ \underline{b}]$

$x_1 + 2x_2 - 1x_5 = -5 \rightarrow x_1 = -5 - 2x_2 + 1 \cdot x_5$

$x_3 = -3 \qquad x_3 = -3$

$x_4 + x_5 = 2 \qquad x_4 = 2 - x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$