Sec. 2.7 Linear Transformations.

<u>Defn</u> function

$f: S_1 \to S_2$, $f(v)$ is a unique element in S_2 for each $v \in S_1$.

S_1: domain
S_2: codomain

$f(v)$: image of v.

Defn: range of f is the set of all images of f.

range of $f = \{ f(v) \in S_2 \mid v \in S_1 \}$.

Ex. $T_A: \mathbb{R}^2 \to \mathbb{R}^3$ $T_A(\underline{x}) = A\underline{x}$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$.

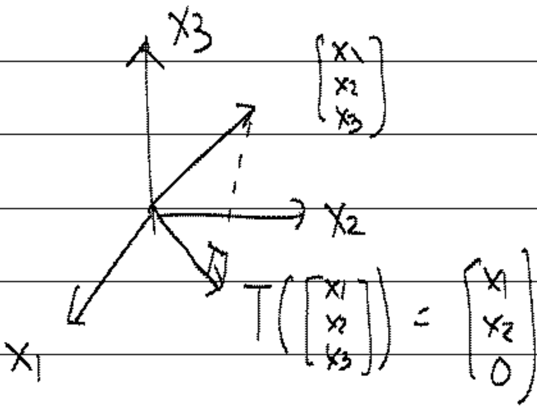
$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}.$$

Defn: Matrix transformation induced by A $(m \times n)$

$T_A: \mathbb{R}^n \to \mathbb{R}^m$ $T_A(\underline{x}) = A\underline{x}$.

Ex. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ $T_A\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$.

orthogonal projection onto the xy plane.



- More examples in HW exercises, rotation, reflection, dilation are matrix transformations.

Thm 2.7 (i) $T_A(\underline{u} + \underline{v}) = T_A(\underline{u}) + T_A(\underline{v})$.

(ii) $T_A(c\underline{u}) = c\, T_A(\underline{u})$, c is scalar.

<u>Defn</u> Linear transformation (L.T.)

$T : \mathbb{R}^n \to \mathbb{R}^m$ is called a L.T. if it satisfies,

(a) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$ for any $\underline{u}, \underline{v} \in \mathbb{R}^n$

(b) $T(c\underline{u}) = c\, T(\underline{u})$, for any scalar c and $\underline{u} \in \mathbb{R}^n$.

- If T is a L.T., then we say T is linear.

- All matrix transformations are linear.
- identity transformation: $I: \mathbb{R}^n \to \mathbb{R}^n \quad I(\underline{x}) = \underline{x}$.
- Zero transformation: $\theta: \mathbb{R}^n \to \mathbb{R}^m \quad \theta(\underline{x}) = \underline{0}$ ← $m \times 1$.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \end{bmatrix}$.

(a) $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 2(u_1+v_1) - (u_2+v_2) \\ (u_1+v_1) \end{bmatrix}$

$= \begin{bmatrix} 2u_1 - u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} 2v_1 - v_2 \\ v_1 \end{bmatrix} = T(\underline{u}) + T(\underline{v})$.

(b) $T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 2cu_1 - cu_2 \\ cu_1 \end{bmatrix} = c\begin{bmatrix} 2u_1 - u_2 \\ u_1 \end{bmatrix} = cT(\underline{u})$

T is linear.

Ex. $T: \mathbb{R} \to \mathbb{R} \quad T(x) = x+1$.

$T(x+y) = x+y+1 \neq x+1+y+1 = T(x) + T(y)$.

T is NOT linear.

Thm 2.8. $T: \mathbb{R}^n \to \mathbb{R}^m$ linear. The following are true.

(a) $T(\underline{0}) = \underline{0}$. (b) $T(-\underline{u}) = -T(\underline{u})$.

(c) $T(\underline{u}-\underline{v}) = T(\underline{u}) - T(\underline{v})$ (d) $T(a\underline{u}+b\underline{v}) = a\,T(\underline{u}) + b\,T(\underline{v})$

for any scalars a, b and any $\underline{u}, \underline{v} \in \mathbb{R}^n$.

- (d) can be extended to $T(c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k)$

$$= c_1 T(\underline{u}_1) + c_2 T(\underline{u}_2) + \cdots + c_k T(\underline{u}_k).$$

Thm 2.9. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then there exists a unique $m \times n$ matrix $A = [T(\underline{e}_1)\ T(\underline{e}_2)\ \ldots\ T(\underline{e}_n)]$ such that

$$T(\underline{x}) = A\underline{x} \quad \text{for each } \underline{x} \in \mathbb{R}^n.$$

i.e. T is equal to the matrix transformation induced by A

$$T = T_A$$

$\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$: standard vectors in $\mathbb{R}^n$.

Proof: For any $\underline{v} \in \mathbb{R}^n$, let us compute

$$T(\underline{v}) = T(v_1 \underline{e}_1 + v_2 \underline{e}_2 + \cdots + v_n \underline{e}_n)$$

$$= v_1 T(\underline{e}_1) + v_2 T(\underline{e}_2) + \cdots + v_n T(\underline{e}_n)$$

$$= \underbrace{\begin{bmatrix} \underbrace{T(\underline{e}_1)}_{m\times 1} & \underbrace{T(\underline{e}_2)}_{m\times 1} & \cdots & T(\underline{e}_n) \end{bmatrix}}_{m\times n} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \overset{m\times n}{A} \underline{v}.$$

Suppose there is another B such that $T(\underline{x}) = B\underline{x}$ for any $\underline{x} \in \mathbb{R}^n$

$$T(\underline{x}) = A\underline{x} = B\underline{x} \quad \text{for any } \underline{x} \in \mathbb{R}^n.$$

From Thm 1.3, we conclude that $A = B$.

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- The $m\times n$ matrix $A = \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & \cdots & T(\underline{e}_n) \end{bmatrix}$ is called the standard matrix of T.

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$. Find A.

$T(\underline{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$T(\underline{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$T(\underline{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Sec. 2.8.

$$\left[\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n\right]$$
$$\|$$

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear with standard matrix A $m \times n$.

$\underline{w}$ is in range of T iff there is a $\underline{v} \in \mathbb{R}^n$ such that $T(\underline{v}) = \underline{w}$

i.e. $A\underline{v} = \underline{w}$.

iff $A\underline{x} = \underline{w}$ is consistent iff $\underline{w}$ is a l.c. of $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n$

iff $\underline{w} \in \text{Span}\{\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n\}$.

- range of $T = \text{Span}\{\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n\}$.

Defn. $T: \mathbb{R}^n \to \mathbb{R}^m$ linear is said to be onto if

range of $T = \mathbb{R}^m$.

Thm 2.10. The following are equivalent.

(a) T is onto. (b) $\text{Span}\{\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n\} = \mathbb{R}^m$.

(c) $\text{rank } A = m$.

(b) $\Leftrightarrow$ (c) from Thm 1.6.

Ex. $T: \mathbb{R}^3 \to \mathbb{R}^3$ with $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 5 & 10 \end{bmatrix}$. Is T onto?

$$A \xrightarrow{G.E.} R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank } A = 2 \neq 3.$$

$\Rightarrow$ T is NOT onto.

Ex. $T: \mathbb{R}^{99} \to \mathbb{R}^{100}$ $A: 100 \times 99$.

rank $A \le 99 \neq 100$ T is not onto.

Defn $T: \mathbb{R}^n \to \mathbb{R}^m$ linear is said to be one-to-one if every pair of distinct vectors in $\mathbb{R}^n$ have distinct images in $\mathbb{R}^m$.

That is, if $\underline{u} \neq \underline{v}$, then $T(\underline{u}) \neq T(\underline{v})$.

or equivalent if $T(\underline{u}) = T(\underline{v})$, then $\underline{u} = \underline{v}$.

Ex. $T: \mathbb{R} \to \mathbb{R}$ $T(x) = \frac{1}{2}x$.

$$\left.\begin{array}{l} \text{(i) if } x \neq y, \text{ then } T(x) = \tfrac{1}{2}x \neq \tfrac{1}{2}y = T(y). \\ \text{(ii) if } T(x) = T(y), \text{ then } \tfrac{1}{2}x = \tfrac{1}{2}y \Rightarrow x = y. \end{array}\right\} T \text{ is one-to-one}$$

Defn. $T: \mathbb{R}^n \to \mathbb{R}^m$ linear with standard matrix A $m \times n$.

Null space of T is the set of all vectors $\underline{v} \in \mathbb{R}^n$ with $T(\underline{v}) = \underline{0}$.

$$T(\underline{v}) = \underline{0} \quad \text{iff} \quad A\underline{v} = \underline{0}.$$

- Null space of T = soln set to $A\underline{x} = \underline{0}$.
- Notice that the zero vector $\underline{0} \in \mathbb{R}^n$ is always in the null space of T.

Claim: T is one-to-one iff null space of $T = \{\underline{0}\}$.

i.e. $A\underline{x} = \underline{0}$ has only zero soln.

Proof: Suppose T is one-to-one,

For any $\underline{v} \neq \underline{0}$ in $\mathbb{R}^n$, $T(\underline{v}) \neq T(\underline{0}) = \underline{0}$.

$$T(\underline{v}) \neq \underline{0} \Rightarrow \underline{v} \notin \text{null space of } T.$$

Suppose null space of $T = \{\underline{0}\}$, i.e. $A\underline{x} = \underline{0}$ has only zero soln.

if $T(\underline{u}) = T(\underline{v})$, then. $T(\underline{u}) - T(\underline{v}) = \underline{0}$.

$\Rightarrow T(\underline{u}-\underline{v}) = \underline{0} \Rightarrow \underline{u}-\underline{v} \in$ Null space of T.

$\underline{u}-\underline{v} = \underline{0} \Rightarrow \underline{u} = \underline{v}$. $\Rightarrow T$ is one-to-one

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$T: \mathbb{R}^3 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$. Is T one-to-one?

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -3 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $A\underline{x} = \underline{0}$.

$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $A \xrightarrow{G.E} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow A\underline{x} = \underline{0} \text{ has nonzero soln}$$

$\Rightarrow T$ is not one-to-one.

From thm 2.11, rank $A = 2 \neq 3 \leftarrow$ n dimension of domain.

Thm 2.11 The following are equivalent. $T: \mathbb{R}^n \to \mathbb{R}^m$.

A : $m \times n$

(a) T is one-to-one.

(b) $A\underline{x} = \underline{0}$ has only zero solution

(c). cols of A are l. indep.

(d) rank $A = n$.

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear

onto? rank $A = m$?

one-to-one?

rank $A = n$?

Composite of L.T's.

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear with standard matrix A $m \times n$.

$U: \mathbb{R}^m \to \mathbb{R}^p$ . . . . B $p \times m$.

$UT: \mathbb{R}^n \to \mathbb{R}^p \quad UT(\underline{x}) = U(T(\underline{x}))$.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^3 \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ x_1-x_2 \\ x_2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$U: \mathbb{R}^3 \to \mathbb{R}^2 \quad U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2+x_3 \\ x_2 \end{bmatrix}. \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

$BA = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$

$UT: \mathbb{R}^2 \to \mathbb{R}^2 . \quad UT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = U\left(T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

$= U\left(\begin{bmatrix} x_1+x_2 \\ x_1-x_2 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)+(x_1-x_2)+x_2 \\ (x_1-x_2) \end{bmatrix}$

$= \begin{bmatrix} 2x_1+x_2 \\ x_1-x_2 \end{bmatrix}. \xrightarrow{} \text{standard matrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

Thm 2.12. UT is also linear. Moreover its standard is given by BA.

Pf. linearity (exercise). $UT(\underline{x}) = U(T(\underline{x})) = U(A\underline{x})$.

$= B(A\underline{x}). = (BA)\underline{x}$

- a function f is invertible iff f is one-to-one and onto.

- L.T. T is invertible iff T is one-to-one and onto.

$T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff rank $A = n$

standard matrix A $n \times n$.

Thm 2.13. $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible iff its standard matrix A is invertible.

Moreover $T^{-1} = T_{A^{-1}}$, i.e. $T^{-1}(\underline{y}) = A^{-1}\underline{x}$.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 5x_2 \end{bmatrix}$. Find T^{-1}.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}. \qquad A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

$$T^{-1}(\underline{x}) = T_{A^{-1}}(\underline{x}) = A^{-1}\underline{x} = \begin{bmatrix} -5x_1 + 2x_2 \\ 3x_1 - x_2 \end{bmatrix}.$$

Exercise. Sec. 2.7. 31, 32, 70, 90, 101, 102.

Sec. 2.8. T&F, 23, 29, 31, 38, 39, 69~74

88~92.

(e) $A\underline{x} = -2\underline{b} - 6.58\,\underline{a}_2$. if $\underline{v}$ is a sol'n to

$A\underline{x} = \underline{b}$,

$A\underline{x} = -2\underline{b}$. then $-2\underline{v}$ is a soln to

$A\underline{x} = -2\underline{b}$

$A(\underline{e}_2) = \underline{a}_2 \Rightarrow A(-6.58\,\underline{e}_1) = -6.58\,\underline{a}_2$.

So $-2\underline{v} + (-6.58\,\underline{e}_2)$ is a soln to

$A\underline{x} = -2\underline{b} - 6.58\,\underline{e}_2$.

$$A \sim R = \begin{bmatrix} 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) $A' = [\underline{a}_2\ \underline{a}_1\ \underline{a}_3\ \underline{a}_4\ \underline{a}_5]$. $R' = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(b) $[I_4\ A]$.

(c). $\underline{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $\underline{a}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\underline{a}_3 = -1 \cdot \underline{a}_1 + 2\,\underline{a}_2 . \rightsquigarrow \underline{a}_2 = \cdots$$

$$\underline{a}_5 = 2\,\underline{a}_1 + 2\,\underline{a}_2 + \underline{a}_4$$

Span of cols of $AB \subset$ Span of cols of A.

$$B = \begin{bmatrix} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_p \end{bmatrix} \quad AB = \begin{bmatrix} A\underline{b}_1 & A\underline{b}_2 & \cdots & A\underline{b}_p \end{bmatrix}$$

If $\underline{v}$ is a l.c. of $A\underline{b}_1, \ldots, A\underline{b}_p$

$$\underline{b}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix}$$

$$\underline{v} = c_1 A\underline{b}_1 + c_2 A\underline{b}_2 + \cdots + c_p A\underline{b}_p$$

$$= c_1 (b_{11}\underline{a}_1 + \cdots + b_{m1}\underline{a}_m) + c_2 (b_{12}\underline{a}_1 + \cdots + b_{m2}\underline{a}_m)$$

$$+ \cdots + c_p (b_{1p}\underline{a}_1 + \cdots + b_{mp}\underline{a}_m).$$

$$= d_1\underline{a}_1 + d_2\underline{a}_2 + \cdots + d_m\underline{a}_m$$