- $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ linearly dependent / independent (l. dep / indep.)

$$c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k = \underline{0}.$$

- Define $A = [\underline{u}_1 \ \underline{u}_2 \cdots \underline{u}_k]$ Look at $A\underline{x} = \underline{0}$.
- S is l dep iff $A\underline{x} = \underline{0}$ has a nonzero soln.

(a)

Thm 1.9. $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ is l. dep iff $\underline{u}_1 = \underline{0}$ or some $\underline{u}_i$ is a l. c. of $\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_{i-1}$. (b)

Pf ($\Leftarrow$) trivial.

($\Rightarrow$) Suppose S is l. dep, i.e. there are scalars $c_1, \cdots, c_k$ (not all zero) such that $c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_k\underline{u}_k = \underline{0}$. — ①

(i) if $\underline{u}_1 = \underline{0}$, done.

(ii). if $\underline{u}_1 \neq \underline{0}$, let c_i be the last nonzero scalar

I.e. $c_i \neq 0$ but $c_{i+1} = 0, \cdots, c_k = 0$.

Then ① can be rewritten as

$$c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_i\underline{u}_i = \underline{0}, \quad c_i \neq 0.$$

Notice that $i \neq 1$. $\because$ if $i = 1$, $c_1\underline{u}_1 = \underline{0} \Rightarrow \underline{u}_1 = \underline{0}$.

$$\underline{u}_i = -\frac{1}{c_i}\left(c_1\underline{u}_1 + \cdots + c_{i-1}\underline{u}_{i-1} \right)$$ #.

Some properties of l. dep & l indep sets.

$S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ subset of $\mathbb{R}^n$.

(1) From Thm1.9, S is l. indep iff $\underline{u}_1 \neq \underline{0}$ and no $\underline{u}_i$ is a l. c. of its preceding vectors.

(2) Suppose S is l indep and $\underline{v} \notin \text{Span } S$.
Then $S' = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k, \underline{v}\}$ is l. indep.

Pf $\underline{u}_1 \neq \underline{0}$ and no vector in S' is a l. c of its preceding vectors.

(3) Every subset of $\mathbb{R}^n$ having more than n vectors is l. dep.

$A = [\underline{u}_1, \underline{u}_2 \cdots \underline{u}_k]$ $n \times k$, if $k > n$, then nullity $A > 0$.

Sec. 1.7. 30, 38, 42, 49, 58, 93, 97, 99.

Review Exercise T&F, 72, 73, 74.

Ch 2. Matrices and Linear Transformations.

Sec. 2.1. Matrix Multiplications.

Defn. A: $m\times n$ $\quad B$: $n\times p$ $\quad B = [\underline{b}_1 \ \underline{b}_2 \ \cdots \ \underline{b}_p]$.

$$AB = [\underset{m\times 1}{A\underline{b}_1} \ \ A\underline{b}_2 \ \cdots \ A\underline{b}_p]. \quad m\times p.$$

If B' is $n\times q$, then form $[B \ B']$. $n\times(p+q)$.

$$A[B \ B'] = [AB \ \ AB'].$$

Ex. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}.$$

① $AB \neq BA$. ② $AB = \underline{\underline{0}}$ does not imply $A = \underline{\underline{0}}$ or $B = \underline{\underline{0}}$

- $(A_1 A_2 \cdots A_k)^T = A_k^T A_{k-1}^T \cdots A_2^T A_1^T$.

Sec. 2.3. Invertibility (square $n \times n$)

<u>Defn</u>. An $n\times n$ matrix A is said to be invertible if there is an $n\times n$ matrix B such that $AB = I_n$ and $BA = I_n$.

B is called the inverse of A, denoted by A^{-1}.

Claim. The inverse of A is unique.

Pf. Suppose both B and C are inverses of A.

$$\left.\begin{array}{l} BAC = B(AC) = B\cdot I_n = B. \\ BAC = (BA)C = I_n \cdot C = C \end{array}\right\} \quad B = C = BAC.$$

\#

. When an $m\times m$ matrix A is invertible, then $A\underline{x} = \underline{b}$ has a unique soln and it is given by $\underline{x} = A^{-1}\underline{b}$.

Thm 2.3 Let A, B be $n\times n$ and invertible. Then the following are true.

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(b) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(c) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Pf (a) $A(A^{-1}) = I_n$ and $(A^{-1})A = I_n$.

(b) $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = I_n$.

$(AB)B^{-1}A^{-1} = \dots - - - = I_n$.

(c) $(A^{-1})^T(A^T) = (A\cdot A^{-1})^T = I^T = I$.

$(A^T)(A^{-1})^T = (A^{-1}\cdot A)^T = I^T = I$. #

Elementary Matrices. (el. matrices)

Defn An n×n matrix E is called an el. matrix if it can be obtained by performing one e.r.o on I_n.

$$I_3 \overset{r_2\leftrightarrow r_3}{\sim} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = E_1,\quad I_3 \xrightarrow[k\neq 0]{kr_2} \begin{bmatrix}1&0&0\\0&k&0\\0&0&1\end{bmatrix} = E_2,\quad I_3 \xrightarrow{cr_1+r_3\to r_3} \begin{bmatrix}1&0&0\\0&1&0\\c&0&1\end{bmatrix} = E_3.$$

- el. matrices are invertible and their inverses are el. matrices.

$$E_1^{-1} = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}.\quad E_2^{-1} = \begin{bmatrix}1&0&0\\0&1/k&0\\0&0&1\end{bmatrix}.\quad E_3^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\-c&0&1\end{bmatrix}.$$

$$A = \begin{bmatrix}1&2\\4&3\\5&6\end{bmatrix}$$

$E_1 A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 4 & 3 \end{bmatrix} \xleftarrow{r_2 \leftrightarrow r_3} A$, $E_2 A = \begin{bmatrix} 1 & 2 \\ 4k & 3k \\ 5 & 6 \end{bmatrix} \xleftarrow{kr_2} A$, $E_3 A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ c+5 & 2c+6 \end{bmatrix} \xleftarrow{cr_1 + r_3 \to r_3} A$

- Multiplying an matrix A by an el. matrix is the same as applying the corresponding e.r.o. on A.

Gaussian Elimination

$$A \xrightarrow{e.r.o\ 1} A_1 \xrightarrow{e.r.o.\ 2} A_2 \longrightarrow \cdots \xrightarrow{e.r.o.\ k} R \quad \text{rref.}$$

$$A_1 = E_1 A \qquad A_2 = E_2 A_1 = E_2 \cdot E_1 A.$$

$$R = \underbrace{E_k E_{k-1} \cdots E_2 E_1}_{P} A \quad \text{———} \quad ①$$

Thm 2.3. Let R be the rref of an $m \times n$ matrix A.
Then there exists an $m \times m$ invertible matrix P such that $PA = R$.

- R is unique. but P may not be unique.
  (P83 Sec 2.4.)
- $\overset{m \times m}{P}$ is unique iff. rank $A = m$.

How to find P?

if we replace $A = I_n$, in ①, we have

$$P = E_k E_{k-1} \cdots E_1 I_m .$$

$$I_m \xrightarrow{e.r.o1} \; - - - - \; \xrightarrow{e.r.ok} P$$

Claim Let $[R \ \underline{c}]$ be the rref of $[A \ \underline{b}]$.

Then $A\underline{x} = \underline{b}$ is equivalent to $R\underline{x} = \underline{c}$.

Pf. From Thm 2.3, there is an $m \times m$ invertible P s.t. $P[A \ \underline{b}] = [R \ \underline{c}]$

① $PA = R$ ③ $A = P^{-1}R$.

② $P\underline{b} = \underline{c}$. ④ $\underline{b} = P^{-1}\underline{c}$

Suppose $\underline{x} = \underline{v}$ is a soln to $A\underline{x} = \underline{b}$, i.e. $A\underline{v} = \underline{b}$. — ⑤

$R\underline{v} \overset{①}{=} PA\underline{v} \overset{⑤}{=} P\underline{b} \overset{②}{=} \underline{c} \Rightarrow \underline{x} = \underline{v}$ is also a soln to $R\underline{x} = \underline{c}$.

Suppose $\underline{x} = \underline{v}$ is a soln to $R\underline{x} = \underline{c}$, i.e. $R\underline{v} = \underline{c}$ — ⑥

$A\underline{v} \overset{③}{=} P^{-1}R\underline{v} \overset{⑥}{=} P^{-1}\underline{c} \overset{④}{=} \underline{b} \Rightarrow \underline{x} = \underline{v}$ is also a soln to $A\underline{x} = \underline{b}$ #

- More generally, let Q be an $m \times m$ invertible matrix.

let $A' = QA$ and $\underline{b}' = Q\underline{b}$. Then

$$A'\underline{x} = \underline{b}' \text{ is equiv. to } A\underline{x} = \underline{b}.$$

$$[A' \ b'] = Q[A \ \ \underline{b}].$$

Ex.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 & 3 \\ 2 & 4 & -3 & 2 & 0 \\ -3 & -6 & 2 & 0 & 3 \end{bmatrix} \xrightarrow{\text{G.E.}} R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\enclose{circle}{\underline{a}_1} \quad \underline{a}_2 \quad \enclose{circle}{\underline{a}_3} \quad \enclose{circle}{\underline{a}_4} \quad \underline{a}_5 \qquad\qquad \underline{r}_1 \quad \underline{r}_2 \quad \underline{r}_3 \quad \underline{r}_4 \quad \underline{r}_5 .$$

$$\underline{a}_2 = 2\underline{a}_1 \qquad \enclose{circle}{c_1\underline{a}_1 + \overset{c_3}{\wedge} a_3 = a_4 ? \quad \text{no soln}} \qquad r_2 = 2r_1$$

$$\underline{a}_5 = \underline{a}_4 - \underline{a}_1 \qquad\qquad \underline{r}_5 = \underline{r}_4 - \underline{r}_1$$

$$\enclose{circle}{c_1 a_1 + c_3 a_3 + c_4 a_4 = a_5 ?} \ \text{yes.}$$

- $\underline{a}_2 = 2\underline{a}_1 \Rightarrow -2\underline{a}_1 + \underline{a}_2 + 0\,\underline{a}_3 + 0\,\underline{a}_4 + 0\,\underline{a}_5 = \underline{0}$.

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{0}. \iff R \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{0}.$$

- Because $A\underline{x} = \underline{0}$ and $R\underline{x} = \underline{0}$, we have.

$$A : m \times n \qquad A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underline{0} \quad \text{iff} \quad R \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underline{0}.$$

Column Correspondence Property. (CCP)

$$c_1\underline{a}_1 + c_2\underline{a}_2 + \cdots + c_n\underline{a}_n = \underline{0} \text{ iff } c_1\underline{r}_1 + c_2\underline{r}_2 + \cdots + c_n\underline{r}_n = \underline{0}.$$

Ex. $A \rightsquigarrow R = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Given $\underline{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$a_2 = 2a_1$ $\qquad a_4 = a_3 - a_3$ $\qquad$ find A.

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

Thm 2.4. The following are true for any matrix A.

(a). Pivot columns of A are l. indep.

(b). Each nonpivot column of A can be uniquely written as a l. c. of its preceding pivot columns.

Thm 1.4. Every matrix can be transformed into one and only one rref.

- Sketch of proof in Appendix E.

Quiz 1 March 28, 9:20 ~ 10:10 am

Ch 1, Sec. 2.1, 2.3, 2.4

Sec. 2.4 The inverse of a matrix.

Thm 2.5. An $n \times n$ matrix A is invertible iff its rref is I_n

Proof: Suppose A is invertible

Let us Look at $A\underline{x} = \underline{0}$, its unique soln is $\underline{x} = \underline{0}$.

$\Rightarrow$ no free variable $\Rightarrow$ nullity $A = 0 \Rightarrow$ rank $A = n \Rightarrow \overset{n\times n.}{\overset{\downarrow}{R}} = I_n$.

because it has n pivot cols.

Suppose $R = I_n$.

from Thm 2.3, there is an $n \times n$ invertible P s.t. $PA = R = I_n$.

$$P^{-1}PA = P^{-1}I_n \Rightarrow A = P^{-1}, \text{ is invertible.}$$

An Algorithm for computing A^{-1} $(n \times n)$.

$$\left[\underset{n\times n}{A} \quad \underset{n\times n}{I_n}\right] \xrightarrow{G.E} \left[\underset{n\times n}{R} \quad \underset{n\times n}{B}\right] \text{ rref.}$$

$P[A \ I_n] = [R \ B]$, P is $n \times n$ invertible.

$PA = R$ and $B = P$. ← another method of finding P.

- If $R \neq I_n$, then A is not invertible.
- If $R = I_n$, then A is invertible and its inverse is given by B.

$$PA = I \Rightarrow P^{-1}PA = P^{-1} \Rightarrow A = P^{-1} \Rightarrow A^{-1} = P.$$

Ex. $[A \quad I_3] = \begin{bmatrix} \textcircled{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{bmatrix}$.

$$\xrightarrow[-3r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \begin{bmatrix} \textcircled{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{2r_2+r_3 \to r_3} \begin{bmatrix} \textcircled{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -2 & 1 & 0 \\ 0 & 0 & \textcircled{-1} & -7 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[-3r_3+r_1 \to r_1]{-1r_3 \to r_3} \begin{bmatrix} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{-2r_2+r_1} \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{R=I} \ \underbrace{\begin{matrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{matrix}}_{B} \right] \qquad A^{-1} = B.$$

Thm 1.6 $A_{m\times n}$ rank $A = m$. ...

Thm 1.8. rank $A = n$.

A $n\times n$.

Thm 2.6. A $n\times n$. The following are equivalent.

Thm 2.5.
- (a) A is invertible.
- (b) rref. of A is I_n.

Thm 1.6.
- (c) rank $A = n$
- (d) span of cols of $A = \mathbb{R}^n$.
- (e) $A\underline{x} = \underline{b}$ is consistent for any $\underline{b} \in \mathbb{R}^n$.

Thm 1.8.
- (f) nullity $A = 0$
- (g) cols of A are l. indep.
- (h) the only soln to $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.

(i) there is an $n\times n$ matrix B such that $BA = I_n$.

(j) there is an $n\times n$ matrix C such that $AC = I_n$.

(k) A is a product of el. matrices.

(a) ~ (h) are equivalent.

$$\begin{tikzcd}
 & (i) \arrow[r, Rightarrow] & (h) \\
(a) \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & (j) \arrow[r, Rightarrow] & (e).
\end{tikzcd}$$

from defn of invertibility

$\underline{(i) \Rightarrow (h)}$ (i) says $\exists$ B s.t. $BA = I_n$

let's look $A\underline{x} = \underline{0} \Rightarrow BA\underline{x} = B \cdot \underline{0} \Rightarrow \underline{x} = \underline{0}$.

So $A\underline{x} = \underline{0}$ has only one soln $\underline{x} = \underline{0}$. (h)

(j) ⇒ (e) (j) says $\exists$ C s.t. $AC = I_n$.

For any $\underline{b} \in \mathbb{R}^n$, let's look at $A\underline{x} = \underline{b}$.

Take $\underline{x} = C\underline{b}$, then $A(C\underline{b}) = (AC)\underline{b} = \underline{b}$.

$\underline{x} = C\underline{b}$ is a soln to $A\underline{x} = \underline{b}$ for any $\underline{b} \in \mathbb{R}^n$ (e),

next we will prove (a) ⇒ (k) and (k) ⇒ (a),

(a) says A is invertible ⇒ r.r.e.f. of A is $\underline{I}$

$$A \rightsquigarrow R = I_n.$$

$$E_k \cdots E_1 A = I_n. \Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}.$$

So A is a product of el. matrices. (k).

(k) says A is a product of el. matrices

$\Rightarrow$ A is invertible ($\because$ el. matrices are invertible)

(a).

$\#$

Sec. 2.1. 61, 62, 63, 65, 68.

Sec. 2.3. T&F, 57, 62, 70, 74, 86~89.

Sec. 2.4 T&F, 17, 18, 33, 65~71, 83.

Sec. 2.4. 68~71. properties of rank.

A: $m \times n$.

① rank A^T = rank A.

② For any $\ell \times m$ B and any $n \times p$ C

rank $BA \le$ rank A , rank $AC \le$ rank A.

③. If B is $m \times m$ invertible and C is $n \times n$ invertible

$\operatorname{rank} BA = \operatorname{rank} A$, $\operatorname{rank} AC = \operatorname{rank} A$.