Sec. 7.2. Linear Transformations

Defn Let V and W be vector spaces (VS).

A mapping $T: V \to W$ is linear if it satisfies

(a) $T(u+v) = T(u) + T(v)$ for any $u, v \in V$

(b) $T(cu) = c\,T(u)$ for any $c \in \mathbb{R}$.

Ex 1. $T: M_{m\times n} \to M_{n\times m}$ $T(A) = A^T$.

(1) $T(A_1 + A_2) = (A_1 + A_2)^T = A_1^T + A_2^T = T(A_1) + T(A_2)$

(2) $T(cA_1) = (cA_1)^T = c\,(A_1)^T = c\,T(A_1)$.

T is linear.

Defn D^∞ : set of all ftns from $\mathbb{R}$ to $\mathbb{R}$ that have derivatives of all orders.

e.g. e^x, $\cos x$, $\sin x$, $p(x)$

Defn. $C([a,b])$: set of all continuous ftns defined on $[a, b]$.

Ex. 2. $\mathcal{D}: C^\infty \to C^\infty$ $\quad \mathcal{D}(f) = f'$

(a) $\mathcal{D}(f_1+f_2) = (f_1+f_2)' = f_1' + f_2' = \mathcal{D}(f_1) + \mathcal{D}(f_2)$.

(b) $\mathcal{D}(cf_1) = (cf_1)' = cf_1' = c\,\mathcal{D}(f_1)$ for any $f_1, f_2 \in C^\infty$

any $c \in \mathbb{R}$.

$\mathcal{D}$ is linear.

Ex. 3. $\quad T: C([a,b]) \to \mathbb{R}$ $\quad T(f) = \int_a^b f(t)\,dt$.

(a) $T(f_1+f_2) = \int_a^b (f_1(t)+f_2(t))\,dt = \int_a^b f_1(t)\,dt + \int_a^b f_2(t)\,dt$

$= T(f_1) + T(f_2)$

(b) $T(cf_1) = c\,T(f_1)$. $\qquad$ T is linear.

Ex 4. $\quad T: \mathcal{P}_2 \to \mathbb{R}^3$ $\quad T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ 2p(1) \end{bmatrix}$.

$p(x) = a + bx + cx^2$ $\quad T(a+bx+cx^2) = \begin{bmatrix} a \\ a+b+c \\ 2(a+b+c) \end{bmatrix}$.

T is linear.

Thm 7.4 (Thm 2.8). The following are true

(a) $T(\theta) = \theta$ (b) $T(u-v) = T(u) - T(v)$

(c) $T(-u) = -T(u)$ (d) $T(au+bv) = aT(u) + bT(v)$,

$a, b \in \mathbb{R}$

- Extend (d) to $T(c_1u_1 + c_2u_2 + \cdots + c_ku_k) = c_1T(u_1) + \cdots + c_kT(u_k)$.

$T: V \to W$ linear

Defn Null space of T = set of all vectors $v \in V$ such that $T(v) = 0$.

range of T = set of all images in W.

Defn T is onto if range of $T = W$.

T is one-to-one if any pair of distinct vectors in V have distinct images.

Thm 7.3 (Thm 2.11) T is one-to-one if and only if null space of $T = \{\theta\}$.

Ex 5 $\quad U: M_{m\times n} \to M_{n\times m} \quad U(A) = A^T$.

$A \in$ Null space of U iff $U(A) = O$ iff $A^T = \theta$

iff $A = \theta$

Null space of $U = \{\theta\}$.

range of U: For any $B \in M_{n\times m}$, let us take $A = B^T$.

Then $A \in M_{m\times n}$ and $U(A) = A^T = B$.

range of $U = M_{n\times m}$.

U is one-to-one and onto.

Ex. 6. $\quad D: C^\infty \to C^\infty \quad D(f) = f'$

let $f(t) = c$ constant functions. Then $f'(t) = f_0(t)$.

if $f(t)$ is not a constant ftn, then $f'(t)$ is not $f_0(t)$

Null space = set of all constant ftns. $\neq \{f_0(t)\}$.

$\mathcal{D}$ is not one-to-one.

e.g $\mathcal{D}(t) = \mathcal{D}(t+c) = 1$.

range of $\mathcal{D}$: For any $g(t) \in C^\infty$, let $h(t)$ be its antiderivative (i.e. $h'(t) = g(t)$).

$$\mathcal{D}(h(t)) = h'(t) = g(t).$$

$$\text{range of } \mathcal{D} = C^\infty.$$

- $\mathcal{D}: C^\infty \to C^\infty$ is onto but not one-to-one.

Compare $T: \mathbb{R}^n \to \mathbb{R}^n$ is onto iff it is one-to-one! different

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Ex. 7. $T: \mathbb{P}_2 \to \mathbb{R}^3$ $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ 2p(1) \end{bmatrix}$.

$$T(a+bx+cx^2) = \begin{bmatrix} a \\ a+b+c \\ 2(a+b+c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ iff } \begin{array}{l} a=0 \\ b=-c. \end{array}$$

Null space of T = set of all polynomials of the form $-cx+cx^2$.

Null space of $T = \text{Span}\{-x+x^2\} \neq \{P_0(x)\}$.

T is not one-to-one.

range of T: $T(a+bx+cx^2) = \begin{bmatrix} a \\ a+b+c \\ 2(a+b+c) \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

range of $T = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \neq \mathbb{R}^3$

T is not onto.

Isomorphism.

<u>Defn</u>. a linear transformation $T: V \to W$ is called an isomorphism if T is one-to-one and onto.

- We say V is isomorphic to W.
- We say V and W are isomorphic.

Thm 7.6. Let $T\ V \to W$ be isomorphic.
Then $T^{-1}: W \to V$ is linear and also isomorphic.

Composition of L.T.'s.

Let V, W, Z be VS.

$T: V \to W$ and $U: W \to Z$ linear.

$UT: V \to Z \quad UT(v) = U(T(v))$.

Thm 7.7. UT is linear.

If U and T are isomorphisms, then UT is also an isomorphism.

Ex. $T: P_2 \to \mathbb{R}^3 \quad T(f) = \begin{bmatrix} p(0) \\ p(1) \\ 2p(1) \end{bmatrix}$.

$U: \mathbb{R}^3 \to M_{2\times 2} \quad U\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

$UT: P_2 \to M_{2\times 2}$

$$UT(a+bx+cx^2) = U\left(\begin{bmatrix} a \\ a+b+c \\ 2(a+b+c) \end{bmatrix}\right) = \begin{bmatrix} a & a+b+c \\ a+b+c & 2(a+b+c) \end{bmatrix}$$

Sec. 7.2. 5, 8, 13, 16, 26, 34, 35, 37, 38.

Sec 7.3. Basis.

<u>Defn.</u> A subset S of a vector space V is said to be l. dep.
if there are scalers $c_1, \dots, c_k$, not all zero and vectors
$v_1, \dots, v_k$ in S such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \theta.$$

otherwise S is called l. indep.

Ex. $S = \{ x^2 - 3x + 2,\ 3x^2 - 5x,\ 2x - 3 \}$ $\subseteq P_2(x)$.

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = \theta$$

S is l. dep.

Ex. $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ l. indep.

$a\times$ $b\times$ $c\times$

Ex. $S = \{ e^t, e^{2t}, e^{3t} \}$. $a e^t + b e^{2t} + c e^{3t} = f_0(t)$.

$t = 0$, $a + b + c = 0$ $(\quad)' = f_0'(t)$

$a + 2b + 3c = 0$

$a + 4b + 9c = 0$. $\qquad (\qquad)'' = f_0''(t)$

$\Rightarrow a = 0,\ b = 0,\ c = 0$.

S is l. indep.

Ex. $S = \{1, x, x^2, \cdots, x^n, \cdots\}$. l. indep.

Thm. 7.8. Let V, W be VS. Let $T: V \to W$ be an isomorphism.
Let $\{v_1, v_2, \cdots, v_k\}$ be a l. indep subset of V.
(basis)
Then $\{T(v_1), T(v_2), \cdots, T(v_k)\}$ is a l. indep subset of W.
(basis).

Defn Basis for V

S is a basis for V if (a) S is l. indep.
(b) $\operatorname{Span} S = V$.

Ex. $S = \{1, x, x^2, x^3\}$ is a basis for P_3.

Ex. W = set of all 2×2 matrices with trace equal to 0.

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$ is a basis for W.

Ex. $W = \{ p(x) \in P_3 : p(x) = p(-x) \}$.

$$p(x) = a + bx + cx^2 + dx^3.$$

$p(x) = p(-x)$ iff $a + bx + cx^2 + dx^3 = a - bx + cx^2 - dx^3$

iff $b = 0, d = 0, \quad a, c \in \mathbb{R}$.

W = set of all third degree polynomials of the form $a + cx^2$.

$= \operatorname{Span}\{1, x^2\}$.

$\{1, x^2\}$ is a basis for W.

Ex $\{1, x, \dots, x^n, \dots\}$ is a basis for P.

Thm. 7.9 (Thm 4.5). Let V be a vector space with a finite basis. Then every basis for V is finite and has the same number of vectors.

Defn $\dim V$ = number of vectors in V.

finite dimensional VS / infinite-dim VS.

Ex.1 $\mathcal{B} = \{1, x, x^2, \cdots, x^n\}$ is a basis for P_n.

$\dim P_n = n+1$.

Ex 2 $M_{2\times 2}$ $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$\dim M_{2\times 2} = 4$.

Ex 3 $M_{m\times n}$ $E_{ij} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$ ← (i,j)th entry

$m \times n$

$S = \{E_{11}, E_{12}, \cdots, E_{1n}, E_{21}, E_{22}, \cdots, E_{2n}, \cdots E_{m1}, E_{m2}, \cdots, E_{mn}\}$.

S is a basis for $M_{m\times n}$.

$$\dim M_{m\times n} = mn.$$

Let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be a basis for V.

Then for every $v \in V$, there exist unique scalars, $c_1, \cdots, c_n$ such that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

<u>Defn</u> Coordinate transformation of V relative $\mathcal{B}$.

$$\phi_{\mathcal{B}} : V \to \mathbb{R}^n. \quad \phi_{\mathcal{B}}(v) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- $\phi_{\mathcal{B}}$ is an isomorphism (proof exercise).
- If $\dim V = n$, then V and $\mathbb{R}^n$ are isomorphic.

<u>Ex 1</u> $\phi_{\mathcal{B}} : P_n \to \mathbb{R}^{n+1}$ $\phi_{\mathcal{B}}(a_0 + a_1 x + \cdots + a_n x^n) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$

<u>Ex 2</u> $\phi_{\mathcal{B}} : M_{2\times 2} \to \mathbb{R}^4$ $\phi_{\mathcal{B}}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Ex. 3 $M_{m\times n}$

$$A = a_{11}T_{11} + a_{12}T_{12} + \cdots + a_{mn}T_{mn}.$$

$$\phi_S : M_{m\times n} \to \mathbb{R}^{mn} \quad \phi_S(A) = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \\ a_{21} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Ex $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. $\quad T: \mathbb{R}^n \to \mathbb{R}^m$.

$m\times n$

$$U: M_{m\times n} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), \quad U(A) = T_A$$

$$T_A: \mathbb{R}^n \to \mathbb{R}^m, \quad T_A(v) = Av.$$

- Every L.T has a unique standard matrix.
- U is one-to-one and onto. U is an isomorphism.

Since S is a basis for $M_{m\times n}$, from Thm 7.8, we know that $S' = \{U(E_{11}), \cdots U(E_{1n}), \cdots, U(E_{mn})\}$ is a basis for $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

$$S' = \{T_{E_{11}}, T_{E_{12}}, \cdots T_{E_{1n}}, \cdots, T_{E_{mn}}\}.$$

$$\dim \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = mn.$$

Let $\overset{m\times n}{A}$ be the standard matrix of T.

$$T = a_{11} T_{E_{11}} + a_{12} T_{E_{12}} + \cdots + a_{1n} T_{E_{12}} + \cdots + a_{mn} T_{E_{mn}}.$$

$$\phi_{S'} : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{mn} \quad \phi_{S'}(T) = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Exercise Sec 7.3, 49 ~ 59, 66.