- Subspace V. (i) (ii) (iii)
- $m \times n$ $A = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] = \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \vdots \\ \underline{r}_m \end{bmatrix}$

$\text{Col } A = \text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$

$\text{Row } A = \text{Span}\{\underline{r}_1^T, \underline{r}_2^T, \ldots, \underline{r}_m^T\}$.

$\text{Null } A =$ soln set to $A\underline{x} = \underline{0}$.

- S is a basis for V if (i) $\text{Span } S = V$ (ii) S is l. indep.
- Pivot cols form a basis for Col A.
- Reduction Thm a generating set $S \xrightarrow{\text{reduce}}$ a basis for V.

Claim Any bases of $\mathbb{R}^n$ have exactly n vectors.

Pf. Let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}$ be a basis for $\mathbb{R}^n$.

① S is a generating set for $\mathbb{R}^n$.

Any vector $\underline{b}$ in $\mathbb{R}^n$ is a l.c. of $\underline{v}_1, \ldots, \underline{v}_k$.

Define $A = [\underline{v}_1 \ \underline{v}_2 \ \cdots \ \underline{v}_k]$ $n \times k$.

Then $A\underline{x} = \underline{b}$ is consistent for any $\underline{b} \in \mathbb{R}^n$.

From Thm 1.6, we know that rank $A = n$.

rank $A \le k \Rightarrow n \le k$.

any generating set of $\mathbb{R}^n$ has more than n vectors.

(2) S is l. indep.

. any subset of $\mathbb{R}^n$ having more than n vectors is l. dep.

. So S cannot have more than n vectors.

$\Rightarrow \quad k \le n$.

Combining (1) and (2), we have $k = n$.

Thm 4.4. (Extension Thm).

Let S be a l. indep subset of a nonzero subspace V of $\mathbb{R}^n$. Then S can be extended to a basis for V by inclusion of additional vectors. In particular, every nonzero subspace has a basis.

Pf. Let $S = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k\}$ be a l. indep subset of V.

$\underline{u}_i \in \mathbb{R}^n$.

If $\operatorname{Span} S = V$, then S is a basis for V. DONE.

If $\operatorname{Span} S \neq V$, then there is a vector $\underline{v}_1 \in V$ but $\underline{v}_1 \notin \operatorname{Span} S$

let $S_1 = \{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k, \underline{v}_1\}$.

Then S_1 is a l. indep subset of V. ($\because$ $\underline{u}_1 \neq \underline{0}$ and no vector in S_1 is a l. c. of its preceding vectors).

If $\operatorname{Span} S_1 = V$, then S_1 is a basis for V. DONE.

If $\operatorname{Span} S_1 \neq V$, then ---.. $\underline{v}_2 \in V$ but $\underline{v}_2 \notin \operatorname{Span} S_1$.

let $S_2 = \{\underline{u}_1, \cdots, \underline{u}_k, \underline{v}_1, \underline{v}_2\}$.

Then S_2 is a l. indep subset of V.

$\vdots$

We can continue this process at most n times.

That means $\operatorname{Span} S_\ell = V$ for some $\ell \leq n$.

$S_\ell = \{\underline{u}_1, \cdots, \underline{u}_k, \underline{v}_1, \cdots, \underline{v}_\ell\}$ is a basis for V.

Let V be a nonzero subspace of $\mathbb{R}^n$.

Let $\underline{v}$ be a nonzero vector in V.

Define $S = \{\underline{v}\}$. Then S is a l. indep subset of V.

Then S can be extended to a basis for V.

//

Thm 4.5. Any 2 bases for a nonzero subspace V of $\mathbb{R}^n$ have the same number of vectors.

Pf: Let $S_1 = \{\underline{u}_1, \cdots, \underline{u}_k\}$ and $S_2 = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_p\}$ be two bases for V.

(i) (k) S_1 is a generating set for V.
(p) S_2 is a l. indep subset of V.

We will prove below that $\underline{k \geq p}$.

Define $\underset{n \times k}{A = [\underline{u}_1\ \underline{u}_2\ \cdots\ \underline{u}_k]}$ and $\underset{n \times p}{B = [\underline{v}_1\ \underline{v}_2\ \cdots\ \underline{v}_p]}$.

$\text{Span}\{\underline{u}_1, \cdots, \underline{u}_k\} = V$ and $\underline{v}_i \in V$.

$\Rightarrow$ $\underline{v}_i$ is a l. c. of $\underline{u}_1, \cdots, \underline{u}_k$

$\Rightarrow$ there is a $k \times 1$ vector $\underline{c}_i$ such that $A\underline{c}_i = \underline{v}_i$.

$$[\underline{v}_1 \ \underline{v}_2 \ \cdots \ \underline{v}_p] = A [\underline{c}_1 \ \underline{c}_2 \ \cdots \ \underline{c}_p].$$

$$B = A \, C \leftarrow k \times p.$$

Next we will prove that $\underline{c}_1, \cdots, \underline{c}_p$ are l. indep.

Suppose there are scalars $d_1, d_2, \cdots, d_p$ s.t.

$$d_1 \underline{c}_1 + \cdots + d_p \underline{c}_p = \underline{0}.$$

$$C \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix} = \underline{0} \Rightarrow C \underline{d} = \underline{0}.$$

Look at $B \underline{d} = (AC) \underline{d} = \underline{0}$.

$B\underline{d} = \underline{0}$ implies that $\underline{d} = \underline{0}$ ($\because$ cols of B are l. indep)

$\Rightarrow \underset{k\times 1}{\underline{c}_1}, \cdots, \underset{k\times 1}{\underline{c}_p}$ are l. indep.

$\underline{\underline{p \le k}}$ ($\because$ any subset of $\mathbb{R}^k$ having more than k vectors is l. dep).

(ii) (p) S_2 is a generating set for V. (k) S_1 is a l. indep subset of V. $\Big\}$ Using a similar procedure, we can prove that $p \ge k$.

Combining (i) and (ii), we have $p = k$. #.

Defn. Dimension of V = number of vectors in a basis for V.
denoted by $\dim V$.

Ex $\dim \mathbb{R}^2 = 2$, $\dim \mathbb{R}^n = n$.

Ex. from last week $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$.

Col A : $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ forms a basis for Col A. (from last week).

So $\dim(\mathrm{Col}\, A) = 2$.

Defn. The dimension of the zero subspace is defined as 0.

Midterm exam. Ch1, Ch2, Ch3, Sec. 4.1, 4.2, 4.3.
4/25, 10:20 am ~ 12:00. (no class at 9:20).

Thm 4.6. Let V be a k-dimensional subspace of $\mathbb{R}^n$.
Then any l. indep subset of V has at most k vectors.

Pf By extension thm.

Ex 1 $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 + x_2 + x_4 = 0 \right\}$. Find a basis for V.

If we define $A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \end{array}$, then $V = \text{Null } A$.

$A\underline{x} = \underline{0}$ x_1: basic, $\underbrace{x_2 \quad x_3 \quad x_4}_{\text{free var's.}}$

general soln in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

l. indep vectors.

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for Null A.

$\dim V = 3$

- The vectors in vector form soln of $A\underline{x} = \underline{0}$ form a basis for Null A.

Thm 4.7. Let V be a nonzero subspace of $\mathbb{R}^n$ and $\dim V = k$. Let S be a subset of V having k vectors.

Then S is a basis for V if either S is l. indep or S is a generating set for V.

Pf (by extension thm and reduction thm, see text).

Confirming that a set S is a basis for V.

(1) Is S a subset of V?

(2) Is $\dim V$ = number of vector in S?

(3) Is S l. indep? OR Is S a generating set for V?

Ex. let $A = [1 \ \ 1 \ \ 0 \ \ 1]$ in Ex 1.
$S = \left\{ \underset{\underline{u}_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\underline{u}_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}}, \underset{\underline{u}_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}} \right\}$

Is S a basis for Null A?

(1) $A\underline{u}_1 = 0 \quad \underline{u}_1 \in \text{Null}\,A \qquad A\underline{u}_3 = 0, \ \underline{u}_3 \in \text{Null}\,A.$

$A\underline{u}_2 = 0 \quad \underline{u}_2 \in \text{Null}\,A \qquad S$ is a subset of Null A.

(2) $\dim(\text{Null } A) = 3$ (from Ex 1). = number of vectors in S.

(3) $\begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ rref. $\Rightarrow S$ is l. indep

(1) + (2) + (3) $\Rightarrow S$ is a basis for Null A.

Sec. 4.2. T&F, 16, 24, 76, 77, 78, 81.

Sec. 4.3. The dimension of subspaces associated with a matrix.

rank A = number of nonzero row vectors in rref of A.
= number of pivot cols of A.

pivot cols of A form a basis for Col A.

$\Rightarrow \quad \dim(\text{Col } A) = \text{rank } A$.

nullity A = n − rank A = number of free var's in the soln of $A\underline{x} = \underline{0}$.

$\dim(\text{Null } A) = \text{nullity } A$.

$$\dim(\text{Col } A) + \dim(\text{Null } A) = n.$$

<u>Claim</u> Let $C = BA$. Then Row C is a subset of Row A.
Moreover if B is invertible, then Row C = Row A.

<u>Pf</u>

$$\underset{p\times n}{C} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \vdots \\ \underline{c}_p \end{bmatrix} \qquad \underset{p\times m}{B} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_p \end{bmatrix} \qquad \underset{m\times n}{A} = \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_m \end{bmatrix}$$

$$C = BA \Rightarrow \underline{c}_i = \underline{b}_i A = \begin{bmatrix} b_{i1} & b_{i2} & \cdots & b_{im} \end{bmatrix} \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_m \end{bmatrix}.$$

$$\underline{c}_i = b_{i1}\underline{a}_1 + b_{i2}\underline{a}_2 + \cdots + b_{im}\underline{a}_m.$$

$\underline{c}_i^T$ are l. c. of $\underline{a}_1^T, \underline{a}_2^T, \cdots, \underline{a}_m^T$.

Span$\{\underline{c}_1^T, \underline{c}_2^T, \cdots, \underline{c}_p^T\}$ is a subset of Span$\{\underline{a}_1^T, \underline{a}_2^T, \cdots, \underline{a}_m^T\}$.

Row C is a subset of Row A.

If B is invertible, then $C = BA$ implies $A = B^{-1}C$

Using the previous result, we conclude that Row A is a subset of Row C. $\Rightarrow$ Row C = Row A. #

Thm 4.8. The nonzero rows in the rref of A form a basis for $\text{Row } A$.

Pf. Let R be the rref of A. There exists an invertible P such that $PA = R$.

From previous claim, we have $\text{Row } A = \text{Row } R$.

and the nonzero rows in R are l. indep.

Nonzero rows form a basis for $\text{Row } R = \text{Row } A$. #

$$\dim \text{Row } A = \text{rank } A.$$

Claim $\text{rank } A^T = \text{rank } A$.

$$\text{Row } A^T = \text{Col } A. \text{ or } \text{Row } A = \text{Col } A^T$$

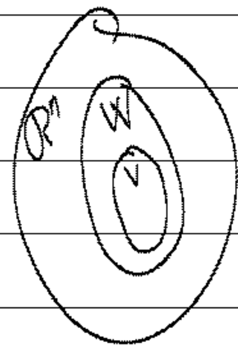
$$\dim(\text{Row } A) = \text{rank } A$$

$$\dim(\text{Col } A^T) = \text{rank } A^T$$

#

$C^T = A^T B^T$. $\text{Col } \underset{\text{Row} C}{\overset{\shortparallel}{C^T}}$ is a subset of $\underset{\text{Row } A}{\overset{\shortparallel}{\text{Col } A^T}}$.

Thm 4.9. If V, W are subspaces of $\mathbb{R}^n$, and V is a subset of W, then $\dim V \leq \dim W$. Moreover, if $\dim V = \dim W$, then $V = W$.



Pf. Let S be a basis for V.

So S is a l. indep subset of W.

By extension thm, S can be extended to a basis for W.

$\Rightarrow \dim W \geq \dim V$.

If $\dim V = \dim W$, by thm 4.7, S is a basis for W.

$\Rightarrow V = W = \text{Span } S$.

Sec. 4.3, T&F, 11, 21, 40, 67, 73~78, 80, 81, 85.

$C = BA$.

Sec. 2.6* LU Decomposition. (NOT in Exam)

lower triangular matrix (l.t.) $\begin{pmatrix} \ddots & 0 & 0 & 0 \\ & \ddots & 0 & 0 \\ & & \ddots & 0 \\ & & & \ddots \end{pmatrix}$

unit lower triangular matrix $\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -5 & \times & \ddots & c_{ij} & 0 \\ & & \times & & \\ \times & & \times & & 1 \end{pmatrix}$

$m \times n$ A During the forward pass of G.E, we use only Type 1 and Type 3 e.r.o's.

Suppose we need only Type 3. e.r.o's in the FP.

$$A \sim \sim \cdots \sim U \text{ ref.}$$

unit l.t. matrix. $E_i = \begin{bmatrix} 1 & & & O \\ 0\,0\,0 & 1 & & \\ 0 & c_{ij} & 0\,0 & \\ 0\,0 & 0 & 0\,0\,0 & 1 \end{bmatrix} \leftarrow \underline{r}_i + c_j \underline{r}_j \to \underline{r}_i$.

$$U = E_l \cdots E_2 E_1 A$$

$E_i^{-1} = \begin{bmatrix} 1 & & O \\ 0 & \ddots & \\ & -c_{ij} & \\ 0 \cdots 0 & 0 & 1 \end{bmatrix}$ unit l.t. matrix

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_\ell^{-1}} U \leftarrow \text{upper triangular matrix.}$$

a product of unit l.t matrices is also unit l.t.

$$A = LU$$

What if Type 1 e.r.o's are needed in FP?

Then A cannot be written as LU.

$$A \xrightarrow{\text{row interchanging}} A' = LU.$$